

Evaluating Medical College Projects with Hamacher Aggregation Operators under the Interval-valued Complex T-Spherical Fuzzy Environment

Mehwish Sarfraz^{1,*}, Rizwan Gul²

¹ Department of Mathematics, Riphah International University Lahore, (Lahore Campus) 5400, Lahore, Pakistan
² Department of Mathematics, Quaid-i-Azam University, Islamabad, 45320, Pakistan

ARTICLE INFO

Article history:

Received 15 January 2025
Received in revised form 20 March 2025
Accepted 23 March 2025
Available online 23 March 2025

Keywords:

Complex Interval-Value T-Spherical Fuzzy Sets; Aggregation Operators; Hamacher Operations; Decision-Making.

ABSTRACT

The study explores the utilization of Hamacher aggregation operators (HAOs) in inscription selection for medical health projects. We employ interval-valued complex T-spherical fuzzy (IVCTSF) information to address the inherent uncertainties in healthcare data. In this paper, we develop the multiple-attribute decision-making (MADM) problems with IVCTSF set information. A few HAOs built on IVCTSF sets are presented in this work. We practice the Hamacher t-norm (HTNM) and Hamacher t-conorm (HTCNM) to characterize certain operational Hamacher operational rules within the context of the IVCTSF sets. We utilize averaging and geometric operations to develop a family of operators for aggregating IVCTSF information, namely IVCTSF Hamacher weighted averaging (IVCTFHWA), IVCTSF Hamacher order weighted averaging (IVCTSFHOWA), IVCTSF Hamacher hybrid weighted averaging (IVCTSFHHWA), IVCTSF Hamacher weighted geometric (IVCTSFHWG), IVCTSF Hamacher ordered weighted geometric (IVCTSFHOWG), and IVCTSF Hamacher hybrid weighted geometric (IVCTSFHHWG) operators. Several noteworthy properties of the developed operators are examined. Besides, an approach to the MADM algorithm is formulated using the proposed operators and is applied to a detailed case study. The case study measures the effectiveness of the proposed algorithm, analyzes the effect of variable parameters on the decision-making procedure, and ensures the stability of ranking results. A comparative analysis is conducted against existing studies to underscore the significance and advantages. Our findings demonstrate the effectiveness of this approach in improving decision-making for healthcare management in complex scenarios.

1. Introduction

The inherent uncertainty and inaccuracy of data in information analysis have long posed major

* Corresponding author.

E-mail address: Mehwish.sarfraz.ms@gmail.com

<https://doi.org/10.31181/msa21202511>

© The Author(s) 2025 | [Creative Commons Attribution 4.0 International License](https://creativecommons.org/licenses/by/4.0/)

challenges for mathematicians. To mitigate these issues, numerous theoretical frameworks have been proposed, each aiming to enhance the precision and reliability of mathematical modeling and data interpretation. Various theoretical frameworks have been developed to address the frequent errors and inconsistencies encountered in data analysis. These theories are characterized by distinct properties, each offering specific advantages and limitations. Among them, Zadeh's fuzzy set (FS) theory has established itself as a foundational approach. This framework effectively addresses key real-world applications, such as navigation, classification, pattern recognition, and numerous domains in computing and engineering. Zadeh introduced the concept of FS to manage uncertainty by representing membership degrees (MDs) as values within the range $[0, 1]$. This mathematical representation enabled researchers to quantify uncertainty in a structured and formalized manner. However, a fundamental limitation of Zadeh's original formulation was its lack of consideration for a non-membership degree (NMD), which restricted its applicability in certain contexts.

Atanassov [1] enhanced Zadeh's impression of an FS and obtained an intuitionistic fuzzy set (IFS) by presenting MD and NMD to address this need. These ideas demonstrated how ambiguous some limitations can be. This idea set a restriction on the aggregate worth of MD and NMD, preventing it from exceeding 1. Yager [2] enhanced this claim by introducing the idea of the Pythagorean fuzzy set (PyFS), which increases the potential for combining the advantages of MD and NMD. Yager [3] also made another outstanding commitment by inventing the q-rung orthopair fuzzy set (q-ROFS) model. Together, the ideas of uncertainty, PyFS, and q-ROFS tackle practical issues like uncertainty and vulnerability.

The largest flaw, however, was their inability to articulate the specifics of approval and rejection because they had degrees. Because human judgment is not confined to a clear-cut "yes" or "no", it might contain a range of replies. Human evaluations also contain a certain amount of tolerance and resistance. Cuong [4] made an effort to account for this peculiarity. According to him, a significant amount of data is lost when ambiguity is not managed, including its summed-up forms of MD and NMD, promotion, and RD. Cuong proposed the idea of a picture fuzzy set (PFS) as a triad that includes MD, NMD, Promotion, and RD with the constraint that the sum of their values should not exceed 1. To diminish this constraint, Mahmood *et al.* [5] lengthened this notion toward a boundary equal by exemplifying strange spherical fuzzy set (SFS) and T-spherical fuzzy set (TSFS) sets. The information from the confused plane is not included in the summarized structures outlined above, according to Ramot *et al.* [6]. By using the baffling numbers rather than the real numbers, Ramot *et al.* [6] considered including the perplexing plane FS and offered the prospect for complex FS (CFS).

1.1. Literature Review

The CFS's notion had pushed the FS to its breaking point, but the CFS could not possibly be concerned about the numerous perplexing numbers in the unit circle. To provide leaders with a massive platform to get rid of the greater data when compared to the CFS, Alkouri *et al.* [7] built the possibility of bewildering uncertainties (CIFs). To promote CIFs, Alkouri and Salleh [7] employed the MD and NMD as a baffling sum after the unit circle on a confusing plane. The MD and NMD of the numbers in the CIFs, however, were represented as a sum of the genuine and fake parts in a unit circle. The issue developed, nevertheless, when leaders selected levels of fantastical and real elements whose aggregate was greater than a unit circle. Ullah *et al.* [8] enclosed more data than the CIFs in their presentation of the complicated PyFSs (CPyFSs) by increasing the number of degrees to the number of their squares.

The most advanced strategy involves choosing the best option from a minor quantity of explicit options created on a sum of frequently at odds with each other criteria. When the aforementioned systems were enhanced, the MADM technique became very popular since the outcomes it generates depend on the most powerful aggregation operators (AOs). Khan *et al.* [9] highlighted the AOs on uncertainty and used them in MADM. Using the PyFSs, Liu and Wang [10] created AOs that were later incorporated into MADM. As part of their contribution to the MADM, Wang *et al.* [11] created AOs for the q-ROFS premise. Garg [12] implemented many AOs in MADM in light of PFSSs. Zhou *et al.* [13] provided AOs for MADM using the TSFS data. There is some outstanding effort on the AOs in [14-16].

A few important functional regulations are required for the creation of these AOs. These regulations rely on a few three-sided criteria to achieve adaptation [17]. Wu *et al.* [18] created Dombi AOs by applying the Dombi t-norm (TNM) and t-conorm (TCNM) in the MADM while taking uncertainties into account. Akram *et al.* [19] framed the Dombi AOs and applied them to PyFS to overcome the MADM issue. Wang and Liu [20] used the Einstein TNM and TCNM to create the AOs for the climate of uncertainty, which were subsequently implemented in MADM. Riaz *et al.* [21] established AOs and provided a beneficial application in the production network of the board by incorporating Einstein TNM and TCNM for the atmosphere of q-ROFSs. Fahmi *et al.* [22] utilized Einstein TNM and TCNM to enhance the AOs for the submission of MADM. Senapati *et al.* [23] suggested the Aczel-Alsina AOs in the context of IFSSs. Yang *et al.* [24] a few long-standing PyFS AOs were promoted in preparation for TCNM and the impending TNM. Several AOs that rely on different additional TNMs and TCNMs are mentioned in [25-26].

The AOs that have a significant influence on the request in MADM are developed using the TNMs and TCNMs that were previously stated. Between these TNMs and TCNMs, the HTNM and HTCNM stand out and have been extensively used by professionals in nearly all of the FS hypothesis models that have been created [27]. Garg [28] formalized AOs and applied HTNM and HTCNM to uncertainty. Wu and Wei [29] also used the HTNM and HTCNM in the formalization of the AOs for the PyFS. For the q-ROFS, Darko and Liang [30] provided a few AOs that used HTNM and HTCNM. To evaluate a project, Ullah *et al.* [31] presented AOs for the TSFS based on HTNM and HTCNM. For IFSSs, the condensed AOs were provided in [32]. PyFS AOs were developed and used by Wu and Wei [33] with the direction of HTNM and HTCNM. Ali *et al.* [34] fostered some AOs under the environment of complex interval-valued PyFS.

It has long been acknowledged that the CTSFS will cover the significant loss of data when we remove data from any real distinctiveness to perform navigation. In particular, it is incredibly possible to extract the most likely facts whenever the human perspective is taken into account. So, employing CTSFS in MADM provides a wonderful opportunity to engage with the outcomes in the MADM. We further noted the importance of HTNM and HTCNM in the work by Klement and Navara [35] who analyzed various TN and TCN kinds, attained a range of ranks, and found the critical ramifications for HTNM and HTCNM.

1.2. Contribution

The key contributions of this study are described as follows:

- i. This paper introduces novel Hamacher operational laws based on IVCTSFSs and applies these operations to develop the IVCTSFHWA, IVCTSFHOWA, IVCTSFHHWA, IVCTSFHWG, IVCTSFHOWG, and IVCTSFHHWG operators.

- ii. This research examines several key mathematical properties of the proposed operators to validate their effectiveness and theoretical robustness.
- iii. The study demonstrates the application of the proposed methods in MAGDM problems using the proposed operators.
- iv. A case study is presented to illustrate the practical benefits of the proposed approach, focusing on the evaluation and government development of a medical college.
- v. To authenticate the efficiency and robustness of the proposed MADM scheme, extensive sensitivity analysis and a comparative evaluation against existing approaches are conducted, highlighting its superiority and resilience.

This article is planned as follows: Section 2 offers several vital notions related to IVCTSFS, score function, and Hamacher operations. In Section 3, we created the Hamacher operational laws for IVCTSFSs. In Section 4, we studied a series of averaging operators for IVCTSFSs, including the IVCTSFHWA, IVCTSFHOWA, and IVCTSFHHWA operators along with their fundamental properties. In Section 5, we constructed the IVCTSFHWG, IVCTSFOWG, and IVCTSFHHWG operators as well as explored their features. Section 6 presents the development of a MADM methodology utilizing the proposed operators. To demonstrate the practical applicability of the developed framework, an illustrative case study is conducted. In Section 7, a comparative and sensitivity analysis is performed to validate the effectiveness and robustness of the proposed approach. Finally, Section 8 outlines the key conclusions and suggests directions for future research.

2. Preliminaries

In this segment, we characterized fundamental ideas connected to IVCTSFS, HTNM, and HTCNM.

Definition 1: [36] The IVCTSFS over a fixed set X can be defined as:

$$I = \left(\begin{array}{l} [\mathbb{m}_a(\mathfrak{X}). e^{2\pi i\vartheta_a(\mathfrak{X})}, \mathbb{u}_a(\mathfrak{X}). e^{2\pi i\beta_a(\mathfrak{X})}] \\ [\mathbb{m}_i(\mathfrak{X}). e^{2\pi i\vartheta_i(\mathfrak{X})}, \mathbb{u}_i(\mathfrak{X}). e^{2\pi i\beta_i(\mathfrak{X})}] \\ [\mathbb{m}_n(\mathfrak{X}). e^{2\pi i\vartheta_n(\mathfrak{X})}, \mathbb{u}_n(\mathfrak{X}). e^{2\pi i\beta_n(\mathfrak{X})}] \end{array} \right); \mathfrak{X} \in X, \quad (1)$$

with $0 \leq \mathbb{m}_a^{\mathbb{Q}}(\mathfrak{X}) + \mathbb{m}_i^{\mathbb{Q}}(\mathfrak{X}) + \mathbb{m}_n^{\mathbb{Q}}(\mathfrak{X}) \leq 1$ and $0 \leq \vartheta_a^{\mathbb{Q}}(\mathfrak{X}) + \vartheta_i^{\mathbb{Q}}(\mathfrak{X}) + \vartheta_n^{\mathbb{Q}}(\mathfrak{X}) \leq 1$ for $\mathbb{Q} \in \mathbb{Z}^+$. The RD can be defined as:

$$\pi(\mathfrak{X}) = \mathbb{m}_h(\mathfrak{X}). e^{2\pi i\vartheta_h(\mathfrak{X})}, \quad (2)$$

where $\mathbb{m}_h(\mathfrak{X}) = \sqrt[\mathbb{Q}]{1 - (\mathbb{m}_a^{\mathbb{Q}}(\mathfrak{X}) + \mathbb{m}_i^{\mathbb{Q}}(\mathfrak{X}) + \mathbb{m}_n^{\mathbb{Q}}(\mathfrak{X}))}$ and $\vartheta_h(\mathfrak{X}) = \sqrt[\mathbb{Q}]{1 - (\vartheta_a^{\mathbb{Q}}(\mathfrak{X}) + \vartheta_i^{\mathbb{Q}}(\mathfrak{X}) + \vartheta_n^{\mathbb{Q}}(\mathfrak{X}))}$.

For simplicity, the triplet

$$([\mathbb{m}_a(\mathfrak{X}). e^{2\pi i\vartheta_a(\mathfrak{X})}, \mathbb{u}_a(\mathfrak{X}). e^{2\pi i\beta_a(\mathfrak{X})}], [\mathbb{m}_i(\mathfrak{X}). e^{2\pi i\vartheta_i(\mathfrak{X})}, \mathbb{u}_i(\mathfrak{X}). e^{2\pi i\beta_i(\mathfrak{X})}], [\mathbb{m}_n(\mathfrak{X}). e^{2\pi i\vartheta_n(\mathfrak{X})}, \mathbb{u}_n(\mathfrak{X}). e^{2\pi i\beta_n(\mathfrak{X})}]), \quad (3)$$

is labeled as an IVCTSFn.

Definition 2: [36] For an IVCTSF I , the score value $\mathbb{S}(I) \in [-1, 1]$ is articulated as:

$$\mathbb{S}(I) = \frac{((\mathbb{m}_a + \vartheta_a + \mathbb{u}_a + \beta_a) - (\mathbb{m}_i + \vartheta_i + \mathbb{u}_i + \beta_i) + (\mathbb{m}_n + \vartheta_n + \mathbb{u}_n + \beta_n))}{6}. \quad (4)$$

For two IVCTSFNs I_1 and I_2 , we can describe the following ordered relations as follows:

- i. When $\mathbb{S}(I_1) < \mathbb{S}(I_2)$, then $I_1 < I_2$.
- ii. When $\mathbb{S}(I_1) > \mathbb{S}(I_2)$, then $I_1 > I_2$.
- iii. When $\mathbb{S}(I_1) = \mathbb{S}(I_2)$, then $I_1 = I_2$.

Definition 3: [27] The HTCNM and HTNM functions can be defined as:

$$T_{hn}(\mathbb{m}, a) = \frac{\mathbb{m} \cdot a}{\mathbb{N} + (1 - \mathbb{N})(\mathbb{m} + a - \mathbb{m}a)}, \quad \mathbb{N} > 0, (\mathbb{m}, a) \in [0, 1]^2, \quad (5)$$

$$T_{hcn}(\mathbb{m}, a) = \frac{\mathbb{m} + a - \mathbb{m}a - (1 - \mathbb{N})\mathbb{m}a}{1 - (1 - \mathbb{N})\mathbb{m}a}, \quad \mathbb{N} > 0, (\mathbb{m}, a) \in [0, 1]^2. \quad (6)$$

3. Hamacher Operational Rules for IVCTSFNs

In this section, we shall provide the concepts of Hamacher operational laws, including sum, product, scalar multiplication, and scalar power which can be applied to aggregate IVCTSFNs. We have also looked at and examined several fundamental properties.

Definition 4: Let A and B be two IVCTSFNs and $B, \mathbb{N} > 0$ be any real numbers. Then, the Hamacher operations for IVCTSFNs are described as:

$$A \oplus B = \left(\begin{array}{l} \left[\begin{array}{l} \sqrt[Q]{\frac{\mathbb{m}_{iA}^Q(x) + \mathbb{m}_{iB}^Q(x) - \mathbb{m}_{iA}^Q(x) \cdot \mathbb{m}_{iB}^Q(x) - (1 - \mathbb{N}) \cdot \mathbb{m}_{iA}^Q(x) \cdot \mathbb{m}_{iB}^Q(x)}{1 - (1 - \mathbb{N}) \cdot \mathbb{m}_{iA}^Q(x) \cdot \mathbb{m}_{iB}^Q(x)}}} \cdot e^{2\pi i \sqrt[Q]{\frac{\vartheta_{iA}^Q(x) + \vartheta_{iB}^Q(x) - \vartheta_{iA}^Q(x) \cdot \vartheta_{iB}^Q(x) - (1 - \mathbb{N}) \cdot \vartheta_{iA}^Q(x) \cdot \vartheta_{iB}^Q(x)}{1 - (1 - \mathbb{N}) \cdot \vartheta_{iA}^Q(x) \cdot \vartheta_{iB}^Q(x)}}} \\ \sqrt[Q]{\frac{\mathbb{w}_{iA}^Q(x) + \mathbb{w}_{iB}^Q(x) - \mathbb{w}_{iA}^Q(x) \cdot \mathbb{w}_{iB}^Q(x) - (1 - \mathbb{N}) \cdot \mathbb{w}_{iA}^Q(x) \cdot \mathbb{w}_{iB}^Q(x)}{1 - (1 - \mathbb{N}) \cdot \mathbb{w}_{iA}^Q(x) \cdot \mathbb{w}_{iB}^Q(x)}}} \cdot e^{2\pi i \sqrt[Q]{\frac{\beta_{iA}^Q(x) + \beta_{iB}^Q(x) - \beta_{iA}^Q(x) \cdot \beta_{iB}^Q(x) - (1 - \mathbb{N}) \cdot \beta_{iA}^Q(x) \cdot \beta_{iB}^Q(x)}{1 - (1 - \mathbb{N}) \cdot \beta_{iA}^Q(x) \cdot \beta_{iB}^Q(x)}}} \end{array} \right], \\ \left[\begin{array}{l} \frac{\mathbb{m}_{iA}(x) \cdot \mathbb{m}_{iB}(x)}{\sqrt[Q]{\mathbb{N} + (1 - \mathbb{N}) (\mathbb{m}_{iA}^Q(x) + \mathbb{m}_{iB}^Q(x) - \mathbb{m}_{iA}^Q(x) \cdot \mathbb{m}_{iB}^Q(x))}} \cdot e^{2\pi i \sqrt[Q]{\frac{\vartheta_{iA}(x) \cdot \vartheta_{iB}(x)}{\mathbb{N} + (1 - \mathbb{N}) (\vartheta_{iA}^Q(x) + \vartheta_{iB}^Q(x) - \vartheta_{iA}^Q(x) \cdot \vartheta_{iB}^Q(x))}}} \\ \frac{\mathbb{w}_{iA}(x) \cdot \mathbb{w}_{iB}(x)}{\sqrt[Q]{\mathbb{N} + (1 - \mathbb{N}) (\mathbb{w}_{iA}^Q(x) + \mathbb{w}_{iB}^Q(x) - \mathbb{w}_{iA}^Q(x) \cdot \mathbb{w}_{iB}^Q(x))}} \cdot e^{2\pi i \sqrt[Q]{\frac{\beta_{iA}(x) \cdot \beta_{iB}(x)}{\mathbb{N} + (1 - \mathbb{N}) (\beta_{iA}^Q(x) + \beta_{iB}^Q(x) - \beta_{iA}^Q(x) \cdot \beta_{iB}^Q(x))}}} \end{array} \right], \\ \left[\begin{array}{l} \frac{\mathbb{m}_{nA}(x) \cdot \mathbb{m}_{nB}(x)}{\sqrt[Q]{\mathbb{N} + (1 - \mathbb{N}) (\mathbb{m}_{nA}^Q(x) + \mathbb{m}_{nB}^Q(x) - \mathbb{m}_{nA}^Q(x) \cdot \mathbb{m}_{nB}^Q(x))}} \cdot e^{2\pi i \sqrt[Q]{\frac{\vartheta_{nA}(x) \cdot \vartheta_{nB}(x)}{\mathbb{N} + (1 - \mathbb{N}) (\vartheta_{nA}^Q(x) + \vartheta_{nB}^Q(x) - \vartheta_{nA}^Q(x) \cdot \vartheta_{nB}^Q(x))}}} \\ \frac{\mathbb{w}_{nA}(x) \cdot \mathbb{w}_{nB}(x)}{\sqrt[Q]{\mathbb{N} + (1 - \mathbb{N}) (\mathbb{w}_{nA}^Q(x) + \mathbb{w}_{nB}^Q(x) - \mathbb{w}_{nA}^Q(x) \cdot \mathbb{w}_{nB}^Q(x))}} \cdot e^{2\pi i \sqrt[Q]{\frac{\beta_{nA}(x) \cdot \beta_{nB}(x)}{\mathbb{N} + (1 - \mathbb{N}) (\beta_{nA}^Q(x) + \beta_{nB}^Q(x) - \beta_{nA}^Q(x) \cdot \beta_{nB}^Q(x))}}} \end{array} \right] \end{array} \right), \quad (7)$$

$$A \otimes B = \left(\begin{array}{c} \left[\frac{\frac{\text{mm}_{a_A}(\mathbf{x}) \cdot \text{mm}_{a_B}(\mathbf{x})}{\sqrt{N+(1-N)(\text{mm}_{a_A}^Q(\mathbf{x}) + \text{mm}_{a_B}^Q(\mathbf{x}) - \text{mm}_{a_A}^Q(\mathbf{x}) \cdot \text{mm}_{a_B}^Q(\mathbf{x}))}} \cdot e^{2\pi i \sqrt{\frac{\vartheta_{a_A}(\mathbf{x}) \cdot \vartheta_{a_B}(\mathbf{x})}{N+(1-N)(\vartheta_{a_A}^Q(\mathbf{x}) + \vartheta_{a_B}^Q(\mathbf{x}) - \vartheta_{a_A}^Q(\mathbf{x}) \cdot \vartheta_{a_B}^Q(\mathbf{x}))}}}{\frac{\text{uu}_{a_A}(\mathbf{x}) \cdot \text{uu}_{a_B}(\mathbf{x})}{\sqrt{N+(1-N)(\text{uu}_{a_A}^Q(\mathbf{x}) + \text{uu}_{a_B}^Q(\mathbf{x}) - \text{uu}_{a_A}^Q(\mathbf{x}) \cdot \text{uu}_{a_B}^Q(\mathbf{x}))}} \cdot e^{2\pi i \sqrt{\frac{\beta_{a_A}(\mathbf{x}) \cdot \beta_{a_B}(\mathbf{x})}{N+(1-N)(\beta_{a_A}^Q(\mathbf{x}) + \beta_{a_B}^Q(\mathbf{x}) - \beta_{a_A}^Q(\mathbf{x}) \cdot \beta_{a_B}^Q(\mathbf{x}))}}}, \right. \\ \left. \left[\frac{\frac{\text{mm}_{i_A}^Q(\mathbf{x}) + \text{mm}_{i_B}^Q(\mathbf{x}) - \text{mm}_{i_A}^Q(\mathbf{x}) \cdot \text{mm}_{i_B}^Q(\mathbf{x}) - (1-N) \cdot \text{mm}_{i_A}^Q(\mathbf{x}) \cdot \text{mm}_{i_B}^Q(\mathbf{x})}{1 - ((1-N) \cdot \text{mm}_{i_A}^Q(\mathbf{x}) \cdot \text{mm}_{i_B}^Q(\mathbf{x}))}} \cdot e^{2\pi i \sqrt{\frac{\vartheta_{i_A}^Q(\mathbf{x}) + \vartheta_{i_B}^Q(\mathbf{x}) - \vartheta_{i_A}^Q(\mathbf{x}) \cdot \vartheta_{i_B}^Q(\mathbf{x}) - (1-N) \cdot \vartheta_{i_A}^Q(\mathbf{x}) \cdot \vartheta_{i_B}^Q(\mathbf{x}2)}}{1 - ((1-N) \cdot \vartheta_{i_A}^Q(\mathbf{x}) \cdot \vartheta_{i_B}^Q(\mathbf{x}))}}}, \right. \\ \left. \frac{\frac{\text{uu}_{i_A}^Q(\mathbf{x}) + \text{uu}_{i_B}^Q(\mathbf{x}) - \text{uu}_{i_A}^Q(\mathbf{x}) \cdot \text{uu}_{i_B}^Q(\mathbf{x}) - (1-N) \cdot \text{uu}_{i_A}^Q(\mathbf{x}) \cdot \text{uu}_{i_B}^Q(\mathbf{x})}{1 - ((1-N) \cdot \text{uu}_{i_A}^Q(\mathbf{x}) \cdot \text{uu}_{i_B}^Q(\mathbf{x}))}} \cdot e^{2\pi i \sqrt{\frac{\beta_{i_A}^Q(\mathbf{x}) + \beta_{i_B}^Q(\mathbf{x}) - \beta_{i_A}^Q(\mathbf{x}) \cdot \beta_{i_B}^Q(\mathbf{x}) - (1-N) \cdot \beta_{i_A}^Q(\mathbf{x}) \cdot \beta_{i_B}^Q(\mathbf{x}2)}}{1 - ((1-N) \cdot \beta_{i_A}^Q(\mathbf{x}) \cdot \beta_{i_B}^Q(\mathbf{x}))}}}, \right. \\ \left. \frac{\frac{\text{mm}_{n_A}^Q(\mathbf{x}) + \text{mm}_{n_B}^Q(\mathbf{x}) - \text{mm}_{n_A}^Q(\mathbf{x}) \cdot \text{mm}_{n_B}^Q(\mathbf{x}) - (1-N) \cdot \text{mm}_{n_A}^Q(\mathbf{x}) \cdot \text{mm}_{n_B}^Q(\mathbf{x})}{1 - ((1-N) \cdot \text{mm}_{n_A}^Q(\mathbf{x}) \cdot \text{mm}_{n_B}^Q(\mathbf{x}))}} \cdot e^{2\pi i \sqrt{\frac{\vartheta_{n_A}^Q(\mathbf{x}) + \vartheta_{n_B}^Q(\mathbf{x}) - \vartheta_{n_A}^Q(\mathbf{x}) \cdot \vartheta_{n_B}^Q(\mathbf{x}) - (1-N) \cdot \vartheta_{n_A}^Q(\mathbf{x}) \cdot \vartheta_{n_B}^Q(\mathbf{x})}}{1 - ((1-N) \cdot \vartheta_{n_A}^Q(\mathbf{x}) \cdot \vartheta_{n_B}^Q(\mathbf{x}))}}}, \right. \\ \left. \frac{\frac{\text{uu}_{n_A}^Q(\mathbf{x}) + \text{uu}_{n_B}^Q(\mathbf{x}) - \text{uu}_{n_A}^Q(\mathbf{x}) \cdot \text{uu}_{n_B}^Q(\mathbf{x}) - (1-N) \cdot \text{uu}_{n_A}^Q(\mathbf{x}) \cdot \text{uu}_{n_B}^Q(\mathbf{x})}{1 - ((1-N) \cdot \text{uu}_{n_A}^Q(\mathbf{x}) \cdot \text{uu}_{n_B}^Q(\mathbf{x}))}} \cdot e^{2\pi i \sqrt{\frac{\beta_{n_A}^Q(\mathbf{x}) + \beta_{n_B}^Q(\mathbf{x}) - \beta_{n_A}^Q(\mathbf{x}) \cdot \beta_{n_B}^Q(\mathbf{x}) - (1-N) \cdot \beta_{n_A}^Q(\mathbf{x}) \cdot \beta_{n_B}^Q(\mathbf{x})}}{1 - ((1-N) \cdot \beta_{n_A}^Q(\mathbf{x}) \cdot \beta_{n_B}^Q(\mathbf{x}))}}} \right] \end{array} \right), \quad (8)$$

$$BA = \left(\begin{array}{c} \left[\frac{\frac{\left(1+(N-1)\text{mm}_{a_A}^Q(\mathbf{x})\right)^B - \left(1-\text{mm}_{a_A}^Q(\mathbf{x})\right)^B}{\left(1+(N-1)\text{mm}_{a_A}^Q(\mathbf{x})\right)^B + (N-1)\left(1-\text{mm}_{a_A}^Q(\mathbf{x})\right)^B} \cdot e^{2\pi i \sqrt{\frac{\left(1+(N-1)\vartheta_{a_A}^Q(\mathbf{x})\right)^B - \left(1-\vartheta_{a_A}^Q(\mathbf{x})\right)^B}{\left(1+(N-1)\vartheta_{a_A}^Q(\mathbf{x})\right)^B + (N-1)\left(1-\vartheta_{a_A}^Q(\mathbf{x})\right)^B}}}, \right. \\ \left. \frac{\frac{\left(1+(N-1)\text{uu}_{a_A}^Q(\mathbf{x})\right)^B - \left(1-\text{uu}_{a_A}^Q(\mathbf{x})\right)^B}{\left(1+(N-1)\text{uu}_{a_A}^Q(\mathbf{x})\right)^B + (N-1)\left(1-\text{uu}_{a_A}^Q(\mathbf{x})\right)^B} \cdot e^{2\pi i \sqrt{\frac{\left(1+(N-1)\beta_{a_A}^Q(\mathbf{x})\right)^B - \left(1-\beta_{a_A}^Q(\mathbf{x})\right)^B}{\left(1+(N-1)\beta_{a_A}^Q(\mathbf{x})\right)^B + (N-1)\left(1-\beta_{a_A}^Q(\mathbf{x})\right)^B}}}, \right. \\ \left. \frac{\frac{\sqrt[N]{\text{mm}_{i_A}(\mathbf{x})}^B}{\sqrt{\left(1+(N-1)\left(1-\text{mm}_{i_A}^Q(\mathbf{x})\right)\right)^B + (N-1)\left(\text{mm}_{i_A}^Q(\mathbf{x})\right)^B}} \cdot e^{2\pi i \sqrt{\frac{\sqrt[N]{\vartheta_{i_A}(\mathbf{x})}^B}{\left(1+(N-1)\left(1-\vartheta_{i_A}^Q(\mathbf{x})\right)\right)^B + (N-1)\left(\vartheta_{i_A}^Q(\mathbf{x})\right)^B}}}, \right. \\ \left. \frac{\frac{\sqrt[N]{\text{uu}_{i_A}(\mathbf{x})}^B}{\sqrt{\left(1+(N-1)\left(1-\text{uu}_{i_A}^Q(\mathbf{x})\right)\right)^B + (N-1)\left(\text{uu}_{i_A}^Q(\mathbf{x})\right)^B}} \cdot e^{2\pi i \sqrt{\frac{\sqrt[N]{\beta_{i_A}(\mathbf{x})}^B}{\left(1+(N-1)\left(1-\beta_{i_A}^Q(\mathbf{x})\right)\right)^B + (N-1)\left(\beta_{i_A}^Q(\mathbf{x})\right)^B}}}, \right. \\ \left. \frac{\frac{\sqrt[N]{\text{mm}_{n_A}(\mathbf{x})}^B}{\sqrt{\left(1+(N-1)\left(1-\text{mm}_{n_A}^Q(\mathbf{x})\right)\right)^B + (N-1)\left(\text{mm}_{n_A}^Q(\mathbf{x})\right)^B}} \cdot e^{2\pi i \sqrt{\frac{\sqrt[N]{\vartheta_{n_A}(\mathbf{x})}^B}{\left(1+(N-1)\left(1-\vartheta_{n_A}^Q(\mathbf{x})\right)\right)^B + (N-1)\left(\vartheta_{n_A}^Q(\mathbf{x})\right)^B}}}, \right. \\ \left. \frac{\frac{\sqrt[N]{\text{uu}_{n_A}(\mathbf{x})}^B}{\sqrt{\left(1+(N-1)\left(1-\text{uu}_{n_A}^Q(\mathbf{x})\right)\right)^B + (N-1)\left(\text{uu}_{n_A}^Q(\mathbf{x})\right)^B}} \cdot e^{2\pi i \sqrt{\frac{\sqrt[N]{\beta_{n_A}(\mathbf{x})}^B}{\left(1+(N-1)\left(1-\beta_{n_A}^Q(\mathbf{x})\right)\right)^B + (N-1)\left(\beta_{n_A}^Q(\mathbf{x})\right)^B}}}, \right] \end{array} \right), \quad (9)$$

$$A^B = \left(\begin{array}{l} \frac{\sqrt[N]{\text{mm}_{\alpha_A}(\mathcal{X})}^B}{\sqrt{\left(1+(N-1)\left(1-\text{mm}_{\alpha_A}^Q(\mathcal{X})\right)^B+(N-1)\left(\text{mm}_{\alpha_A}^Q(\mathcal{X})\right)^B\right)^B}} \cdot e^{2\pi i \frac{\sqrt[N]{\vartheta_{\alpha_A}(\mathcal{X})}^B}{\sqrt{\left(1+(N-1)\left(1-\vartheta_{\alpha_A}^Q(\mathcal{X})\right)^B+(N-1)\left(\vartheta_{\alpha_A}^Q(\mathcal{X})\right)^B}}} \\ \frac{\sqrt[N]{\text{uu}_{\alpha_A}(\mathcal{X})}^B}{\sqrt{\left(1+(N-1)\left(1-\text{uu}_{\alpha_A}^Q(\mathcal{X})\right)^B+(N-1)\left(\text{uu}_{\alpha_A}^Q(\mathcal{X})\right)^B\right)^B}} \cdot e^{2\pi i \frac{\sqrt[N]{\beta_{\alpha_A}(\mathcal{X})}^B}{\sqrt{\left(1+(N-1)\left(1-\beta_{\alpha_A}^Q(\mathcal{X})\right)^B+(N-1)\left(\beta_{\alpha_A}^Q(\mathcal{X})\right)^B}}} \\ \frac{\sqrt{\frac{\left(1+(N-1)\text{mm}_{i_A}^Q(\mathcal{X})\right)^B - \left(1-\text{mm}_{i_A}^Q(\mathcal{X})\right)^B}{\left(1+(N-1)\text{mm}_{i_A}^Q(\mathcal{X})\right)^B + (N-1)\left(1-\text{mm}_{i_A}^Q(\mathcal{X})\right)^B}}}{\sqrt{\left(1+(N-1)\text{mm}_{i_A}^Q(\mathcal{X})\right)^B + (N-1)\left(1-\text{mm}_{i_A}^Q(\mathcal{X})\right)^B}} \cdot e^{2\pi i \frac{\sqrt{\frac{\left(1+(N-1)\vartheta_{i_A}^Q(\mathcal{X})\right)^B - \left(1-\vartheta_{i_A}^Q(\mathcal{X})\right)^B}{\left(1+(N-1)\vartheta_{i_A}^Q(\mathcal{X})\right)^B + (N-1)\left(1-\vartheta_{i_A}^Q(\mathcal{X})\right)^B}}}{\sqrt{\left(1+(N-1)\vartheta_{i_A}^Q(\mathcal{X})\right)^B + (N-1)\left(1-\vartheta_{i_A}^Q(\mathcal{X})\right)^B}}} \\ \frac{\sqrt{\frac{\left(1+(N-1)\text{uu}_{i_A}^Q(\mathcal{X})\right)^B - \left(1-\text{uu}_{i_A}^Q(\mathcal{X})\right)^B}{\left(1+(N-1)\text{uu}_{i_A}^Q(\mathcal{X})\right)^B + (N-1)\left(1-\text{uu}_{i_A}^Q(\mathcal{X})\right)^B}}}{\sqrt{\left(1+(N-1)\text{uu}_{i_A}^Q(\mathcal{X})\right)^B + (N-1)\left(1-\text{uu}_{i_A}^Q(\mathcal{X})\right)^B}} \cdot e^{2\pi i \frac{\sqrt{\frac{\left(1+(N-1)\beta_{i_A}^Q(\mathcal{X})\right)^B - \left(1-\beta_{i_A}^Q(\mathcal{X})\right)^B}{\left(1+(N-1)\beta_{i_A}^Q(\mathcal{X})\right)^B + (N-1)\left(1-\beta_{i_A}^Q(\mathcal{X})\right)^B}}}{\sqrt{\left(1+(N-1)\beta_{i_A}^Q(\mathcal{X})\right)^B + (N-1)\left(1-\beta_{i_A}^Q(\mathcal{X})\right)^B}}} \\ \frac{\sqrt{\frac{\left(1+(N-1)\text{mm}_{n_A}^Q(\mathcal{X})\right)^B - \left(1-\text{mm}_{n_A}^Q(\mathcal{X})\right)^B}{\left(1+(N-1)\text{mm}_{n_A}^Q(\mathcal{X})\right)^B + (N-1)\left(1-\text{mm}_{n_A}^Q(\mathcal{X})\right)^B}}}{\sqrt{\left(1+(N-1)\text{mm}_{n_A}^Q(\mathcal{X})\right)^B + (N-1)\left(1-\text{mm}_{n_A}^Q(\mathcal{X})\right)^B}} \cdot e^{2\pi i \frac{\sqrt{\frac{\left(1+(N-1)\vartheta_{n_A}^Q(\mathcal{X})\right)^B - \left(1-\vartheta_{n_A}^Q(\mathcal{X})\right)^B}{\left(1+(N-1)\vartheta_{n_A}^Q(\mathcal{X})\right)^B + (N-1)\left(1-\vartheta_{n_A}^Q(\mathcal{X})\right)^B}}}{\sqrt{\left(1+(N-1)\vartheta_{n_A}^Q(\mathcal{X})\right)^B + (N-1)\left(1-\vartheta_{n_A}^Q(\mathcal{X})\right)^B}}} \\ \frac{\sqrt{\frac{\left(1+(N-1)\text{uu}_{n_A}^Q(\mathcal{X})\right)^B - \left(1-\text{uu}_{n_A}^Q(\mathcal{X})\right)^B}{\left(1+(N-1)\text{uu}_{n_A}^Q(\mathcal{X})\right)^B + (N-1)\left(1-\text{uu}_{n_A}^Q(\mathcal{X})\right)^B}}}{\sqrt{\left(1+(N-1)\text{uu}_{n_A}^Q(\mathcal{X})\right)^B + (N-1)\left(1-\text{uu}_{n_A}^Q(\mathcal{X})\right)^B}} \cdot e^{2\pi i \frac{\sqrt{\frac{\left(1+(N-1)\beta_{n_A}^Q(\mathcal{X})\right)^B - \left(1-\beta_{n_A}^Q(\mathcal{X})\right)^B}{\left(1+(N-1)\beta_{n_A}^Q(\mathcal{X})\right)^B + (N-1)\left(1-\beta_{n_A}^Q(\mathcal{X})\right)^B}}}{\sqrt{\left(1+(N-1)\beta_{n_A}^Q(\mathcal{X})\right)^B + (N-1)\left(1-\beta_{n_A}^Q(\mathcal{X})\right)^B}}} \end{array} \right) \quad (10)$$

4. Averaging Aggregation Operators of IVCTSFNs

In this part, using Hamacher operational laws, we construct a series of averaging AOs for IVCTSFNs, namely IVCTSFHWA, IVCTSFHOWA, and IVCTSFHHWA operators. We also highlight the desirable characteristics and of these AOs with concrete illustrations. Throughout, this script $\Gamma_{\mathcal{E}} = (\Gamma_1, \Gamma_2, \dots, \Gamma_{\text{mm}})^T$ will denote the weight vector, such that $\Gamma_{\mathcal{E}} > 0$ and $\sum_{\mathcal{E}=1}^{\text{mm}} \Gamma_{\mathcal{E}} = 1$ for $\mathcal{E} = \{1, 2, 3 \dots \text{mm}\}$.

Definition 5: Let $\mathcal{T}_{\mathcal{E}}$ be the collection of IVCTSFNs. Then, the IVCTSFHWA operator is mapping $T^{\text{mm}} \rightarrow T$ defined as:

$$\text{IVCTSFHWA}(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \dots, \mathcal{T}_{\text{mm}}) = \sum_{\mathcal{E}=1}^{\text{mm}} \Gamma_{\mathcal{E}} \mathcal{T}_{\mathcal{E}}. \quad (11)$$

Theorem 1: Let $T_{\mathcal{E}}$ be a group of IVCTSFNs. Then, the aggregated outcome using IVCTSFHWA is still an IVCTSFN, which is postulated as:

$$IVCTSFHWA(\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \dots, \mathcal{J}_{mm}) =$$

$$\left(\begin{array}{l} \left[\begin{array}{l} \frac{\prod_{\mathcal{E}=1}^Q (1+(N-1)mm_{a_{\mathcal{E}}^Q})^{r_{\mathcal{E}}} - \prod_{\mathcal{E}=1}^Q (1-mm_{a_{\mathcal{E}}^Q})^{r_{\mathcal{E}}}}{\sqrt{\prod_{\mathcal{E}=1}^Q (1+(N-1)mm_{a_{\mathcal{E}}^Q})^{r_{\mathcal{E}}} + (N-1)\prod_{\mathcal{E}=1}^Q (1-mm_{a_{\mathcal{E}}^Q})^{r_{\mathcal{E}}}}} \cdot e^{2\pi i \frac{\prod_{\mathcal{E}=1}^Q (1+(N-1)\vartheta_{a_{\mathcal{E}}^Q})^{r_{\mathcal{E}}} - \prod_{\mathcal{E}=1}^Q (1-\vartheta_{a_{\mathcal{E}}^Q})^{r_{\mathcal{E}}}}{\prod_{\mathcal{E}=1}^Q (1+(N-1)\vartheta_{a_{\mathcal{E}}^Q})^{r_{\mathcal{E}}} + (N-1)\prod_{\mathcal{E}=1}^Q (1-\vartheta_{a_{\mathcal{E}}^Q})^{r_{\mathcal{E}}}}} \\ \frac{\prod_{\mathcal{E}=1}^Q (1+(N-1)w_{a_{\mathcal{E}}^Q})^{r_{\mathcal{E}}} - \prod_{\mathcal{E}=1}^Q (1-w_{a_{\mathcal{E}}^Q})^{r_{\mathcal{E}}}}{\sqrt{\prod_{\mathcal{E}=1}^Q (1+(N-1)w_{a_{\mathcal{E}}^Q})^{r_{\mathcal{E}}} + (N-1)\prod_{\mathcal{E}=1}^Q (1-w_{a_{\mathcal{E}}^Q})^{r_{\mathcal{E}}}}} \cdot e^{2\pi i \frac{\prod_{\mathcal{E}=1}^Q (1+(N-1)\beta_{a_{\mathcal{E}}^Q})^{r_{\mathcal{E}}} - \prod_{\mathcal{E}=1}^Q (1-\beta_{a_{\mathcal{E}}^Q})^{r_{\mathcal{E}}}}{\prod_{\mathcal{E}=1}^Q (1+(N-1)\beta_{a_{\mathcal{E}}^Q})^{r_{\mathcal{E}}} + (N-1)\prod_{\mathcal{E}=1}^Q (1-\beta_{a_{\mathcal{E}}^Q})^{r_{\mathcal{E}}}}} \end{array} \right] \\ \left[\begin{array}{l} \frac{\sqrt[Q]{N} \prod_{\mathcal{E}=1}^Q (mm_{i_{\mathcal{E}}})^{r_{\mathcal{E}}}}{\sqrt{\prod_{\mathcal{E}=1}^Q (1+(N-1)(1-mm_{i_{\mathcal{E}}^Q})^{r_{\mathcal{E}}}) + (N-1)\prod_{\mathcal{E}=1}^Q (mm_{i_{\mathcal{E}}^Q})^{r_{\mathcal{E}}}}} \cdot e^{2\pi i \frac{\sqrt[Q]{N} \prod_{\mathcal{E}=1}^Q (\vartheta_{i_{\mathcal{E}}})^{r_{\mathcal{E}}}}{\prod_{\mathcal{E}=1}^Q (1+(N-1)(1-\vartheta_{i_{\mathcal{E}}^Q})^{r_{\mathcal{E}}}) + (N-1)\prod_{\mathcal{E}=1}^Q (\vartheta_{i_{\mathcal{E}}^Q})^{r_{\mathcal{E}}}}} \\ \frac{\sqrt[Q]{N} \prod_{\mathcal{E}=1}^Q (w_{i_{\mathcal{E}}})^{r_{\mathcal{E}}}}{\sqrt{\prod_{\mathcal{E}=1}^Q (1+(N-1)(1-w_{i_{\mathcal{E}}^Q})^{r_{\mathcal{E}}}) + (N-1)\prod_{\mathcal{E}=1}^Q (w_{i_{\mathcal{E}}^Q})^{r_{\mathcal{E}}}}} \cdot e^{2\pi i \frac{\sqrt[Q]{N} \prod_{\mathcal{E}=1}^Q (\beta_{i_{\mathcal{E}}})^{r_{\mathcal{E}}}}{\prod_{\mathcal{E}=1}^Q (1+(N-1)(1-\beta_{i_{\mathcal{E}}^Q})^{r_{\mathcal{E}}}) + (N-1)\prod_{\mathcal{E}=1}^Q (\beta_{i_{\mathcal{E}}^Q})^{r_{\mathcal{E}}}}} \end{array} \right] \\ \left[\begin{array}{l} \frac{\sqrt[Q]{N} \prod_{\mathcal{E}=1}^Q (mm_{n_{\mathcal{E}}})^{r_{\mathcal{E}}}}{\sqrt{\prod_{\mathcal{E}=1}^Q (1+(N-1)(1-mm_{n_{\mathcal{E}}^Q})^{r_{\mathcal{E}}}) + (N-1)\prod_{\mathcal{E}=1}^Q (mm_{n_{\mathcal{E}}^Q})^{r_{\mathcal{E}}}}} \cdot e^{2\pi i \frac{\sqrt[Q]{N} \prod_{\mathcal{E}=1}^Q (\vartheta_{n_{\mathcal{E}}})^{r_{\mathcal{E}}}}{\prod_{\mathcal{E}=1}^Q (1+(N-1)(1-\vartheta_{n_{\mathcal{E}}^Q})^{r_{\mathcal{E}}}) + (N-1)\prod_{\mathcal{E}=1}^Q (\vartheta_{n_{\mathcal{E}}^Q})^{r_{\mathcal{E}}}}} \\ \frac{\sqrt[Q]{N} \prod_{\mathcal{E}=1}^Q (w_{n_{\mathcal{E}}})^{r_{\mathcal{E}}}}{\sqrt{\prod_{\mathcal{E}=1}^Q (1+(N-1)(1-w_{n_{\mathcal{E}}^Q})^{r_{\mathcal{E}}}) + (N-1)\prod_{\mathcal{E}=1}^Q (w_{n_{\mathcal{E}}^Q})^{r_{\mathcal{E}}}}} \cdot e^{2\pi i \frac{\sqrt[Q]{N} \prod_{\mathcal{E}=1}^Q (\beta_{n_{\mathcal{E}}})^{r_{\mathcal{E}}}}{\prod_{\mathcal{E}=1}^Q (1+(N-1)(1-\beta_{n_{\mathcal{E}}^Q})^{r_{\mathcal{E}}}) + (N-1)\prod_{\mathcal{E}=1}^Q (\beta_{n_{\mathcal{E}}^Q})^{r_{\mathcal{E}}}}} \end{array} \right] \end{array} \right) \quad (12)$$

Proof of Theorem 1 is provided Appendix-1.

Theorem 2: For the IVCTSFHWA operator, the following features hold true:

- i. **Idempotency** – If for all $T_{\mathcal{E}} = T$, then:

$$IVCTSFHWA(\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \dots, \mathcal{J}_{mm}) = \mathcal{J}. \quad (13)$$

- ii. **Boundedness** – Assume T^- and T^+ . Then:

$$\mathcal{J}^- \leq IVCTSFHWA(\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \dots, \mathcal{J}_{mm}) \leq \mathcal{J}^+. \quad (14)$$

- iii. **Monotonicity** – Let $\mathcal{J}_{\mathcal{E}}$ and $P_{\mathcal{E}}$ be two collections of IVCTSFNs, such that $\mathcal{J}_{\mathcal{E}} \leq P_{\mathcal{E}}, \forall \mathcal{E}$. Then:

$$IVCTSFHWA(\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \dots, \mathcal{J}_{mm}) \leq IVCTSFHWA(P_1, P_2, P_3, \dots, P_{mm}). \quad (15)$$

Proof of Theorem 2 is straightforward and therefore omitted.

The IVCTSFHWA operator first assigns the weights to IVCTSFNs and then determines their weighted aggregated value. Now, we articulate another operator termed as IVCTSFHOWA operator,

which first organizes the IVCTSFNs in descending order and then allocates weights to their ordered positions. This operator is then implemented to aggregate the ordered weighted IVCTSFNs.

Definition 6: Let $T_{\mathcal{E}}$ be a group of IVCTSFNs. We can define the IVCTSFHOWA operator as:

$$IVCTSFHOWA(\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \dots, \mathcal{J}_{\mathbb{m}}) = \sum_{\mathcal{E}=1}^{\mathbb{L}} \Gamma_{\mathcal{E}} \mathcal{J}_{\sigma(\mathcal{E})}, \tag{16}$$

where $(\sigma(1), \sigma(2), \dots, \sigma(\mathbb{m}))$ is a permutation such that $\mathcal{J}_{\sigma(\mathcal{E}-1)} \geq \mathcal{J}_{\sigma(\mathcal{E})}, \forall \mathcal{E}$.

Theorem 3: The aggregated outcomes of the collection of IVCTSFNs $T_{\mathcal{E}}$ by the IVCTSFHOWA operator is again an IVCTSFN and is postulated as:

$$IVCTSFHOWA(\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \dots, \mathcal{J}_{\mathbb{m}}) = \sum_{\mathcal{E}=1}^{\mathbb{L}} \Gamma_{\mathcal{E}} \mathcal{J}_{\sigma(\mathcal{E})} \tag{17}$$

$$= \left(\begin{array}{l} \left[\begin{array}{l} \frac{\sqrt[\mathbb{Q}]{\frac{\prod_{\mathcal{E}=1}^{\mathbb{m}} (1+(N-1) \theta_{\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}} - \prod_{\mathcal{E}=1}^{\mathbb{m}} (1-\theta_{\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}}}{\prod_{\mathcal{E}=1}^{\mathbb{m}} (1+(N-1) \theta_{\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}} + (N-1) \prod_{\mathcal{E}=1}^{\mathbb{m}} (1-\theta_{\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}}}} \cdot e^{2\pi i \frac{\sqrt[\mathbb{Q}]{\frac{\prod_{\mathcal{E}=1}^{\mathbb{m}} (1+(N-1) \theta_{\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}} - \prod_{\mathcal{E}=1}^{\mathbb{m}} (1-\theta_{\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}}}{\prod_{\mathcal{E}=1}^{\mathbb{m}} (1+(N-1) \theta_{\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}} + (N-1) \prod_{\mathcal{E}=1}^{\mathbb{m}} (1-\theta_{\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}}}}}{\sqrt[\mathbb{Q}]{\frac{\prod_{\mathcal{E}=1}^{\mathbb{m}} (1+(N-1) \beta_{\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}} - \prod_{\mathcal{E}=1}^{\mathbb{m}} (1-\beta_{\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}}}{\prod_{\mathcal{E}=1}^{\mathbb{m}} (1+(N-1) \beta_{\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}} + (N-1) \prod_{\mathcal{E}=1}^{\mathbb{m}} (1-\beta_{\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}}}} \cdot e^{2\pi i \frac{\sqrt[\mathbb{Q}]{\frac{\prod_{\mathcal{E}=1}^{\mathbb{m}} (1+(N-1) \beta_{\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}} - \prod_{\mathcal{E}=1}^{\mathbb{m}} (1-\beta_{\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}}}{\prod_{\mathcal{E}=1}^{\mathbb{m}} (1+(N-1) \beta_{\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}} + (N-1) \prod_{\mathcal{E}=1}^{\mathbb{m}} (1-\beta_{\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}}}} \end{array} \right] \\ \left[\begin{array}{l} \frac{\sqrt[\mathbb{Q}]{\frac{\prod_{\mathcal{E}=1}^{\mathbb{m}} (1+(N-1) \vartheta_{\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}} - \prod_{\mathcal{E}=1}^{\mathbb{m}} (1-\vartheta_{\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}}}{\prod_{\mathcal{E}=1}^{\mathbb{m}} (1+(N-1) \vartheta_{\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}} + (N-1) \prod_{\mathcal{E}=1}^{\mathbb{m}} (1-\vartheta_{\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}}}} \cdot e^{2\pi i \frac{\sqrt[\mathbb{Q}]{\frac{\prod_{\mathcal{E}=1}^{\mathbb{m}} (1+(N-1) \vartheta_{\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}} - \prod_{\mathcal{E}=1}^{\mathbb{m}} (1-\vartheta_{\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}}}{\prod_{\mathcal{E}=1}^{\mathbb{m}} (1+(N-1) \vartheta_{\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}} + (N-1) \prod_{\mathcal{E}=1}^{\mathbb{m}} (1-\vartheta_{\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}}}} \cdot e^{\frac{\sqrt[\mathbb{Q}]{\prod_{\mathcal{E}=1}^{\mathbb{m}} (\vartheta_{i\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}}}}{\sqrt[\mathbb{Q}]{\frac{\prod_{\mathcal{E}=1}^{\mathbb{m}} (1+(N-1) (1-\vartheta_{i\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}}} + (N-1) \prod_{\mathcal{E}=1}^{\mathbb{m}} (\vartheta_{i\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}}}} \cdot e^{2\pi i \frac{\sqrt[\mathbb{Q}]{\prod_{\mathcal{E}=1}^{\mathbb{m}} (\vartheta_{i\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}}}}{\sqrt[\mathbb{Q}]{\frac{\prod_{\mathcal{E}=1}^{\mathbb{m}} (1+(N-1) (1-\vartheta_{i\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}}} + (N-1) \prod_{\mathcal{E}=1}^{\mathbb{m}} (\vartheta_{i\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}}}} \end{array} \right] \\ \left[\begin{array}{l} \frac{\sqrt[\mathbb{Q}]{\frac{\prod_{\mathcal{E}=1}^{\mathbb{m}} (1+(N-1) \vartheta_{\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}} - \prod_{\mathcal{E}=1}^{\mathbb{m}} (1-\vartheta_{\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}}}{\prod_{\mathcal{E}=1}^{\mathbb{m}} (1+(N-1) \vartheta_{\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}} + (N-1) \prod_{\mathcal{E}=1}^{\mathbb{m}} (1-\vartheta_{\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}}}} \cdot e^{2\pi i \frac{\sqrt[\mathbb{Q}]{\frac{\prod_{\mathcal{E}=1}^{\mathbb{m}} (1+(N-1) \beta_{\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}} - \prod_{\mathcal{E}=1}^{\mathbb{m}} (1-\beta_{\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}}}{\prod_{\mathcal{E}=1}^{\mathbb{m}} (1+(N-1) \beta_{\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}} + (N-1) \prod_{\mathcal{E}=1}^{\mathbb{m}} (1-\beta_{\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}}}} \cdot e^{\frac{\sqrt[\mathbb{Q}]{\prod_{\mathcal{E}=1}^{\mathbb{m}} (\beta_{i\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}}}}{\sqrt[\mathbb{Q}]{\frac{\prod_{\mathcal{E}=1}^{\mathbb{m}} (1+(N-1) (1-\beta_{i\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}}} + (N-1) \prod_{\mathcal{E}=1}^{\mathbb{m}} (\beta_{i\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}}}} \cdot e^{2\pi i \frac{\sqrt[\mathbb{Q}]{\prod_{\mathcal{E}=1}^{\mathbb{m}} (\beta_{i\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}}}}{\sqrt[\mathbb{Q}]{\frac{\prod_{\mathcal{E}=1}^{\mathbb{m}} (1+(N-1) (1-\beta_{i\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}}} + (N-1) \prod_{\mathcal{E}=1}^{\mathbb{m}} (\beta_{i\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}}}} \end{array} \right] \\ \left[\begin{array}{l} \frac{\sqrt[\mathbb{Q}]{\frac{\prod_{\mathcal{E}=1}^{\mathbb{m}} (1+(N-1) \vartheta_{\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}} - \prod_{\mathcal{E}=1}^{\mathbb{m}} (1-\vartheta_{\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}}}{\prod_{\mathcal{E}=1}^{\mathbb{m}} (1+(N-1) \vartheta_{\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}} + (N-1) \prod_{\mathcal{E}=1}^{\mathbb{m}} (1-\vartheta_{\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}}}} \cdot e^{2\pi i \frac{\sqrt[\mathbb{Q}]{\frac{\prod_{\mathcal{E}=1}^{\mathbb{m}} (1+(N-1) \vartheta_{n\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}} - \prod_{\mathcal{E}=1}^{\mathbb{m}} (1-\vartheta_{n\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}}}{\prod_{\mathcal{E}=1}^{\mathbb{m}} (1+(N-1) \vartheta_{n\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}} + (N-1) \prod_{\mathcal{E}=1}^{\mathbb{m}} (1-\vartheta_{n\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}}}} \cdot e^{\frac{\sqrt[\mathbb{Q}]{\prod_{\mathcal{E}=1}^{\mathbb{m}} (\vartheta_{n\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}}}}{\sqrt[\mathbb{Q}]{\frac{\prod_{\mathcal{E}=1}^{\mathbb{m}} (1+(N-1) (1-\vartheta_{n\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}}} + (N-1) \prod_{\mathcal{E}=1}^{\mathbb{m}} (\vartheta_{n\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}}}} \cdot e^{2\pi i \frac{\sqrt[\mathbb{Q}]{\prod_{\mathcal{E}=1}^{\mathbb{m}} (\vartheta_{n\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}}}}{\sqrt[\mathbb{Q}]{\frac{\prod_{\mathcal{E}=1}^{\mathbb{m}} (1+(N-1) (1-\vartheta_{n\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}}} + (N-1) \prod_{\mathcal{E}=1}^{\mathbb{m}} (\vartheta_{n\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}}}} \end{array} \right] \\ \left[\begin{array}{l} \frac{\sqrt[\mathbb{Q}]{\frac{\prod_{\mathcal{E}=1}^{\mathbb{m}} (1+(N-1) \vartheta_{\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}} - \prod_{\mathcal{E}=1}^{\mathbb{m}} (1-\vartheta_{\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}}}{\prod_{\mathcal{E}=1}^{\mathbb{m}} (1+(N-1) \vartheta_{\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}} + (N-1) \prod_{\mathcal{E}=1}^{\mathbb{m}} (1-\vartheta_{\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}}}} \cdot e^{2\pi i \frac{\sqrt[\mathbb{Q}]{\frac{\prod_{\mathcal{E}=1}^{\mathbb{m}} (1+(N-1) \beta_{n\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}} - \prod_{\mathcal{E}=1}^{\mathbb{m}} (1-\beta_{n\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}}}{\prod_{\mathcal{E}=1}^{\mathbb{m}} (1+(N-1) \beta_{n\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}} + (N-1) \prod_{\mathcal{E}=1}^{\mathbb{m}} (1-\beta_{n\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}}}} \cdot e^{\frac{\sqrt[\mathbb{Q}]{\prod_{\mathcal{E}=1}^{\mathbb{m}} (\beta_{n\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}}}}{\sqrt[\mathbb{Q}]{\frac{\prod_{\mathcal{E}=1}^{\mathbb{m}} (1+(N-1) (1-\beta_{n\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}}} + (N-1) \prod_{\mathcal{E}=1}^{\mathbb{m}} (\beta_{n\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}}}} \cdot e^{2\pi i \frac{\sqrt[\mathbb{Q}]{\prod_{\mathcal{E}=1}^{\mathbb{m}} (\beta_{n\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}}}}{\sqrt[\mathbb{Q}]{\frac{\prod_{\mathcal{E}=1}^{\mathbb{m}} (1+(N-1) (1-\beta_{n\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}}} + (N-1) \prod_{\mathcal{E}=1}^{\mathbb{m}} (\beta_{n\sigma(\mathcal{E})}^{\mathbb{Q}})^{\Gamma_{\mathcal{E}}}}} \end{array} \right] \end{array} \right)$$

Proof of Theorem 3 is analogous to proof of Theorem 1.

The IVCTSFHOWA operator possesses the axioms of idempotency, monotonicity, and boundedness. The IVCTSFHWA operator weighs only IVCTSFNs, while the IVCTSFHOWA operator weighs just the ordered positions of IVCTSFNs. Consequently, weights express different aspects of the IVCTSFHWA and LDFHOWA operators. However, one of the operators, as well as the other operators, considers just one of them. To circumvent this shortcoming, we interpret the IVCTSFHHA

5. Geometric Aggregation Operators of IVCTSFNs

In this section, using Hamacher operational laws, we establish a class of geometric AOs for IVCTSFNs, namely the IVCTSFHWG, IVCTSFHOWG, and IVCTSFHHWG operators and studied some of their basic features.

Definition 8: For a group of IVCTSFNs $T_{\mathcal{E}}$ the IVCTSFHWG operator is portrayed as:

$$IVCTSFHWG(\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \dots, \mathcal{J}_{mm}) = \sum_{\mathcal{E}=1}^l \mathcal{J}_{\mathcal{E}}^{r_{\mathcal{E}}} \tag{20}$$

Theorem 5: Consider $T_{\mathcal{E}}$ be IVCTSFNs. The IVCTSFHWG operator is described as:

$$IVCTSFHWG(\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \dots, \mathcal{J}_{mm}) = \left(\begin{array}{l} \left[\frac{\sqrt[Q]{\prod_{\mathcal{E}=1}^l (1+(N-1)(1-\vartheta_{a_{\mathcal{E}}})^{r_{\mathcal{E}}})}}{\sqrt[Q]{\prod_{\mathcal{E}=1}^l (1+(N-1)(1-\vartheta_{a_{\mathcal{E}}})^{r_{\mathcal{E}}}) + (N-1)\prod_{\mathcal{E}=1}^l (\vartheta_{a_{\mathcal{E}}})^{r_{\mathcal{E}}}}} \right] \cdot e^{2\pi i \frac{\sqrt[Q]{\prod_{\mathcal{E}=1}^l (\vartheta_{a_{\mathcal{E}}})^{r_{\mathcal{E}}}}}{\sqrt[Q]{\prod_{\mathcal{E}=1}^l (1+(N-1)(1-\vartheta_{a_{\mathcal{E}}})^{r_{\mathcal{E}}}) + (N-1)\prod_{\mathcal{E}=1}^l (\vartheta_{a_{\mathcal{E}}})^{r_{\mathcal{E}}}}}}, \\ \left[\frac{\sqrt[Q]{\prod_{\mathcal{E}=1}^l (1+(N-1)(1-\varpi_{a_{\mathcal{E}}})^{r_{\mathcal{E}}})}}{\sqrt[Q]{\prod_{\mathcal{E}=1}^l (1+(N-1)(1-\varpi_{a_{\mathcal{E}}})^{r_{\mathcal{E}}}) + (N-1)\prod_{\mathcal{E}=1}^l (\varpi_{a_{\mathcal{E}}})^{r_{\mathcal{E}}}}} \right] \cdot e^{2\pi i \frac{\sqrt[Q]{\prod_{\mathcal{E}=1}^l (\varpi_{a_{\mathcal{E}}})^{r_{\mathcal{E}}}}}{\sqrt[Q]{\prod_{\mathcal{E}=1}^l (1+(N-1)(1-\varpi_{a_{\mathcal{E}}})^{r_{\mathcal{E}}}) + (N-1)\prod_{\mathcal{E}=1}^l (\varpi_{a_{\mathcal{E}}})^{r_{\mathcal{E}}}}}, \\ \left[\frac{\sqrt[Q]{\prod_{\mathcal{E}=1}^l (1+(N-1)\vartheta_{i_{\mathcal{E}}}^{r_{\mathcal{E}}}) - \prod_{\mathcal{E}=1}^l (1-\vartheta_{i_{\mathcal{E}}}^{r_{\mathcal{E}}})}}{\sqrt[Q]{\prod_{\mathcal{E}=1}^l (1+(N-1)\vartheta_{i_{\mathcal{E}}}^{r_{\mathcal{E}}}) + (N-1)\prod_{\mathcal{E}=1}^l (1-\vartheta_{i_{\mathcal{E}}}^{r_{\mathcal{E}}})}} \right] \cdot e^{2\pi i \frac{\sqrt[Q]{\prod_{\mathcal{E}=1}^l (\vartheta_{i_{\mathcal{E}}})^{r_{\mathcal{E}}}}}{\sqrt[Q]{\prod_{\mathcal{E}=1}^l (1+(N-1)\vartheta_{i_{\mathcal{E}}}^{r_{\mathcal{E}}}) + (N-1)\prod_{\mathcal{E}=1}^l (1-\vartheta_{i_{\mathcal{E}}}^{r_{\mathcal{E}}})}}}, \\ \left[\frac{\sqrt[Q]{\prod_{\mathcal{E}=1}^l (1+(N-1)\varpi_{i_{\mathcal{E}}}^{r_{\mathcal{E}}}) - \prod_{\mathcal{E}=1}^l (1-\varpi_{i_{\mathcal{E}}}^{r_{\mathcal{E}}})}}{\sqrt[Q]{\prod_{\mathcal{E}=1}^l (1+(N-1)\varpi_{i_{\mathcal{E}}}^{r_{\mathcal{E}}}) + (N-1)\prod_{\mathcal{E}=1}^l (1-\varpi_{i_{\mathcal{E}}}^{r_{\mathcal{E}}})}} \right] \cdot e^{2\pi i \frac{\sqrt[Q]{\prod_{\mathcal{E}=1}^l (\varpi_{i_{\mathcal{E}}})^{r_{\mathcal{E}}}}}{\sqrt[Q]{\prod_{\mathcal{E}=1}^l (1+(N-1)\varpi_{i_{\mathcal{E}}}^{r_{\mathcal{E}}}) + (N-1)\prod_{\mathcal{E}=1}^l (1-\varpi_{i_{\mathcal{E}}}^{r_{\mathcal{E}}})}}}, \\ \left[\frac{\sqrt[Q]{\prod_{\mathcal{E}=1}^l (1+(N-1)\vartheta_{n_{\mathcal{E}}}^{r_{\mathcal{E}}}) - \prod_{\mathcal{E}=1}^l (1-\vartheta_{n_{\mathcal{E}}}^{r_{\mathcal{E}}})}}{\sqrt[Q]{\prod_{\mathcal{E}=1}^l (1+(N-1)\vartheta_{n_{\mathcal{E}}}^{r_{\mathcal{E}}}) + (N-1)\prod_{\mathcal{E}=1}^l (1-\vartheta_{n_{\mathcal{E}}}^{r_{\mathcal{E}}})}} \right] \cdot e^{2\pi i \frac{\sqrt[Q]{\prod_{\mathcal{E}=1}^l (\vartheta_{n_{\mathcal{E}}})^{r_{\mathcal{E}}}}}{\sqrt[Q]{\prod_{\mathcal{E}=1}^l (1+(N-1)\vartheta_{n_{\mathcal{E}}}^{r_{\mathcal{E}}}) + (N-1)\prod_{\mathcal{E}=1}^l (1-\vartheta_{n_{\mathcal{E}}}^{r_{\mathcal{E}}})}}}, \\ \left[\frac{\sqrt[Q]{\prod_{\mathcal{E}=1}^l (1+(N-1)\varpi_{n_{\mathcal{E}}}^{r_{\mathcal{E}}}) - \prod_{\mathcal{E}=1}^l (1-\varpi_{n_{\mathcal{E}}}^{r_{\mathcal{E}}})}}{\sqrt[Q]{\prod_{\mathcal{E}=1}^l (1+(N-1)\varpi_{n_{\mathcal{E}}}^{r_{\mathcal{E}}}) + (N-1)\prod_{\mathcal{E}=1}^l (1-\varpi_{n_{\mathcal{E}}}^{r_{\mathcal{E}}})}} \right] \cdot e^{2\pi i \frac{\sqrt[Q]{\prod_{\mathcal{E}=1}^l (\varpi_{n_{\mathcal{E}}})^{r_{\mathcal{E}}}}}{\sqrt[Q]{\prod_{\mathcal{E}=1}^l (1+(N-1)\varpi_{n_{\mathcal{E}}}^{r_{\mathcal{E}}}) + (N-1)\prod_{\mathcal{E}=1}^l (1-\varpi_{n_{\mathcal{E}}}^{r_{\mathcal{E}}})}}} \end{array} \right) \tag{21}$$

Proof of Theorem 5 is analogous to proof of Theorem 1.

Definition 9: Consider $T_{\mathcal{E}}$ be a collection of IVCTSFNs. Then, the IVCTSFHOWG operator is postulated as:

$$IVCTSFHOWG(\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \dots, \mathcal{J}_{mm}) = \sum_{\mathcal{E}=1}^l \mathcal{J}_{\sigma(\mathcal{E})}^{r_{\mathcal{E}}} \tag{22}$$

where $(\sigma(1), \sigma(2), \dots, \sigma(mm))$ is a permutation such that $\mathcal{J}_{\sigma(\mathcal{E}-1)} \geq \mathcal{J}_{\sigma(\mathcal{E})}, \forall \mathcal{E}$.

Theorem 6: Consider $T_{\mathcal{E}}$ be a family of IVCTSFNs. Then, the IVCTSFHOWG operator is given as:

6. Application of MADM

In this segment, we implemented the proposed operators to construct a MADM scheme under the environment of the IVCTSFS setting. Let $B = \{B_1, B_2, \dots, B_k\}$ be an assemblage of alternatives and $v = \{v_1, v_2, \dots, v_\varepsilon\}$ be the set attributes. Assume that $\Gamma_\varepsilon = (\Gamma_1, \Gamma_2, \dots, \Gamma_m)^T$ be the associated weight vector of attributes, such that $\Gamma_\varepsilon > 0$ and $\sum_1^m \Gamma_\varepsilon = 1$.

Presume that $D_{k \times \varepsilon} = (T)_{k \times \varepsilon} = \left(\begin{matrix} [\mathbb{m}_a(\mathfrak{X}). e^{2\pi i \vartheta_a(\mathfrak{X})}, \mathbb{w}_a(\mathfrak{X}). e^{2\pi i \beta_a(\mathfrak{X})}] \\ [\mathbb{m}_i(\mathfrak{X}). e^{2\pi i \vartheta_i(\mathfrak{X})}, \mathbb{w}_i(\mathfrak{X}). e^{2\pi i \beta_i(\mathfrak{X})}] \\ [\mathbb{m}_n(\mathfrak{X}). e^{2\pi i \vartheta_n(\mathfrak{X})}, \mathbb{w}_n(\mathfrak{X}). e^{2\pi i \beta_n(\mathfrak{X})}] \end{matrix} \right)_{k \times \varepsilon}$ be the decision

matrix of IVCTSFNs provided by an expert to evaluate the given alternatives under given attributes. Considering these observations, the systematic process of designed MADM scheme is outlined as follows:

Step 1 – Structure the given information in the decision matrix form of IVCTSFNs.

Step 2 – Evaluate the normalized matrix $\hat{N} = (\hat{T})_{k \times \varepsilon}$ according to the following transformation:

$$\hat{T} = \begin{cases} T & \text{for benefit attribute} \\ T^c & \text{for cost attribute} \end{cases} \quad (26)$$

Step 3 – Utilize the proposed AOs to aggregate data in the form of IVCTSFNs.

Step 4 – Compute the score values of each alternative using the following formula:

$$S(T) = \frac{((\mathbb{m}_a + \vartheta_a + \mathbb{w}_a + \beta_a) - (\mathbb{m}_i + \vartheta_i + \mathbb{w}_i + \beta_i) - (\mathbb{m}_n + \vartheta_n + \mathbb{w}_n + \beta_n))}{6} \quad (27)$$

Step 5 – Rank the alternatives based on their score values. The greater the score value, the better the alternative is.

6.1. Numerical Example

This example helps us apply the proposed way to deal with the MADM issue. The government made a medical college project in the city and selected a suitable place for this project. Medical colleges are essential institutions that play a multifaceted role in training healthcare professionals, advancing medical research, providing healthcare services, and contributing to the overall health and well-being of communities. They are foundational elements of healthcare systems, supporting the delivery of quality care and the pursuit of medical knowledge. The benefits of medical colleges are:

- i. *Training healthcare professionals* – To educate and train doctors, nurses, and allied health professionals, ensuring a skilled workforce for healthcare delivery.
- ii. *Advancing medical research* – Researching to improve medical knowledge, treatments, and patient outcomes.
- iii. *Addressing healthcare disparities* – Alleviating healthcare disparities by producing professionals attuned to diverse community needs.
- iv. *Providing specialized medical services* – Offering specialized medical care and treatments through associated hospitals and clinics.

- v. Meeting community healthcare needs – Medical colleges can engage in international collaborations, sharing expertise, research findings, and best practices with the global healthcare community.

In essence, medical colleges are critical for training, research, and healthcare delivery, ensuring a healthy and skilled workforce for the benefit of society. Ultimately, the suitability of an area for a medical college is a complex decision that takes into account the unique characteristics and needs of the region, to improve healthcare services and education for the benefit of the local population. The government wants to select a city to make medical college due to some basic's cause. This city's population is larger than other cities. The government chooses four cities for this project. The branch of wellbeing needs to choose one city $B_{\bar{E}}$ ($1 \leq \bar{E} \leq 4$) in view of certain criteria (v_1) , (v_2) , (v_3) , and (v_4) . The weights are $\Gamma_{\bar{E}} = (0.1, 0.2, 0.3, 0.4)^T$.

Step 1 – The data provided by an expert is organized in the from of IVCTFNs (Table 1).

Table 1
 Decision matrix

	v_1	v_2
B_1	$([0.3, 0.4]e^{2\pi i[0.2,0.3]}, [0.2, 0.3]e^{2\pi i[0.3,0.4]}, [0.2, 0.4]e^{2\pi i[0.2,0.4]})$	$([0.3, 0.5]e^{2\pi i[0.2,0.3]}, [0.32, 0.35]e^{2\pi i[0.22,0.25]}, [0.1, 0.3]e^{2\pi i[0.2,0.3]})$
B_2	$([0.2, 0.3]e^{2\pi i[0.1,0.4]}, [0.1, 0.3]e^{2\pi i[0.4,0.5]}, [0.3, 0.4]e^{2\pi i[0.2,0.3]})$	$([0.3, 0.4]e^{2\pi i[0.3,0.4]}, [0.22, 0.34]e^{2\pi i[0.32,0.46]}, [0.2, 0.4]e^{2\pi i[0.3,0.4]})$
B_3	$([0.1, 0.4]e^{2\pi i[0.2,0.4]}, [0.3, 0.4]e^{2\pi i[0.1,0.3]}, [0.2, 0.3]e^{2\pi i[0.1,0.2]})$	$([0.2, 0.4]e^{2\pi i[0.4,0.5]}, [0.34, 0.35]e^{2\pi i[0.25,0.44]}, [0.2, 0.4]e^{2\pi i[0.2,0.4]})$
B_4	$([0.2, 0.6]e^{2\pi i[0.3,0.5]}, [0.4, 0.5]e^{2\pi i[0.2,0.4]}, [0.3, 0.4]e^{2\pi i[0.3,0.4]})$	$([0.1, 0.2]e^{2\pi i[0.3,0.4]}, [0.21, 0.25]e^{2\pi i[0.17,0.45]}, [0.3, 0.4]e^{2\pi i[0.1,0.2]})$
	v_3	v_4
B_1	$([0.1, 0.2]e^{2\pi i[0.2,0.3]}, [0.22, 0.35]e^{2\pi i[0.24,0.28]}, [0.2, 0.3]e^{2\pi i[0.3,0.4]})$	$([0.1, 0.3]e^{2\pi i[0.2,0.3]}, [0.24, 0.25]e^{2\pi i[0.22,0.25]}, [0.3, 0.4]e^{2\pi i[0.1,0.4]})$
B_2	$([0.3, 0.4]e^{2\pi i[0.3,0.4]}, [0.24, 0.33]e^{2\pi i[0.34,0.35]}, [0.3, 0.4]e^{2\pi i[0.2,0.3]})$	$([0.2, 0.3]e^{2\pi i[0.3,0.4]}, [0.22, 0.35]e^{2\pi i[0.32,0.35]}, [0.3, 0.5]e^{2\pi i[0.2,0.5]})$
B_3	$([0.4, 0.5]e^{2\pi i[0.1,0.2]}, [0.17, 0.35]e^{2\pi i[0.22,0.25]}, [0.1, 0.2]e^{2\pi i[0.2,0.4]})$	$([0.3, 0.4]e^{2\pi i[0.4,0.5]}, [0.21, 0.25]e^{2\pi i[0.23,0.34]}, [0.2, 0.3]e^{2\pi i[0.2,0.4]})$
B_4	$([0.2, 0.4]e^{2\pi i[0.1,0.3]}, [0.19, 0.25]e^{2\pi i[0.21,0.27]}, [0.2, 0.3]e^{2\pi i[0.1,0.2]})$	$([0.4, 0.5]e^{2\pi i[0.4,0.5]}, [0.16, 0.22]e^{2\pi i[0.22,0.33]}, [0.2, 0.4]e^{2\pi i[0.3,0.4]})$

Step 2 – There is no need for normalization since there is no cost attribute.

Step 3 – The aggregated values (fixing $\mathbb{Q} = 4, \mathbb{N} = 3$) via the IVCTSFHWA and IVCTSFHWG operators are displayed in Table 2.

Table 2
 Aggregated values

	IVCTSFHWA	IVCTSFHWG
B_1	$([0.0893, 0.1279]e^{2\pi i[0.0895,0.1186]}, [0.0314, 0.039]e^{2\pi i[0.0301,0.0355]}, [0.0261, 0.0456]e^{2\pi i[0.0218,0.0497]})$	$([0.0178, 0.0402]e^{2\pi i[0.0255,0.0391]}, [0.1036, 0.1223]e^{2\pi i[0.1016,0.1138]}, [0.0871, 0.1308]e^{2\pi i[0.0921,0.1383]})$
B_2	$([0.1059, 0.1330]e^{2\pi i[0.1052,0.14445]}, [0.0266, 0.0441]e^{2\pi i[0.0439,0.0511]}, [0.0359, 0.0573]e^{2\pi i[0.0278,0.0511]})$	$([0.0316, 0.0456]e^{2\pi i[0.0347,0.0527]}, [0.0876, 0.1276]e^{2\pi i[0.1287,0.1437]}, [0.1108, 0.1526]e^{2\pi i[0.0976,0.1419]})$
B_3	$([0.10561, 0.1516]e^{2\pi i[0.1127,0.144445]}, [0.0291, 0.0409]e^{2\pi i[0.0271,0.0424]}, [0.1, 0.2]e^{2\pi i[0.2,0.4]})$	$([0.0349, 0.0562]e^{2\pi i[0.0321,0.0495]}, [0.1044, 0.1264]e^{2\pi i[0.0892,0.1270]}, [0.0812, 0.1174]e^{2\pi i[0.0810,0.1358]})$
B_4	$([0.0960, 0.1463]e^{2\pi i[0.1112,0.1489]}, [0.0252, 0.0335]e^{2\pi i[0.0260,0.0446]}, [0.0206, 0.0484]e^{2\pi i[0.0221,0.037]})$	$([0.0295, 0.0530]e^{2\pi i[0.0316,0.0544]}, [0.0971, 0.11301]e^{2\pi i[0.0890,0.1324]}, [0.1032, 0.1371]e^{2\pi i[0.0846,0.1155]})$

Step 4 – The score values are calculated in Table 3.

Table 3

Score values

	IVCTSFHWA	IVCTSFHWG
$S(B_1)$	0.0509	-0.3807
$S(B_2)$	0.0423	-0.3839
$S(B_3)$	0.0818	-0.3489
$S(B_4)$	0.0806	-0.3515

Step 5 – Based on the score values, the outcomes through the IVCTSFHWA and IVCTSFHWG operators are obtained as $B_3 > B_4 > B_1 > B_2$ and $B_3 > B_4 > B_1 > B_2$, respectively.

7. Discussion and Comparison

The present section conducts a comparative sensitivity analysis and comparison of the designed AOs to confirm the validity and supremacy of our recommended technique and operators.

7.1. Sensitivity Analysis

Depending on the decision-maker choices, multiple values might be allocated to the Hamacher parameter $\Gamma_{\mathcal{E}}$. To scrutinize the influence of the parameter $\Gamma_{\mathcal{E}}$ on the recommended MADM strategy, we accomplish an analysis using numerous values of $\Gamma_{\mathcal{E}}$. The overall score values and the ranking results utilizing the IVCTSFHWA and IVCTSFHWG operators related to these values of $\Gamma_{\mathcal{E}}$ are summarized in Table 4 and Table 5, respectively.

Table 4

Impact of $\Gamma_{\mathcal{E}}$ under the IVCTSFHWA operator

Inputs of $\Gamma_{\mathcal{E}}$	Score values				Rankings
	B_1	B_2	B_3	B_4	
2	-0.2365	-0.2457	-0.2203	-0.2226	$B_3 > B_4 > B_2 > B_1$
4	-0.1878	-0.0901	-0.1662	-0.1682	$B_3 > B_4 > B_1 > B_2$
5	-0.1693	-0.178	-0.1459	-0.1478	$B_4 > B_3 > B_1 > B_2$
7	-0.1396	-0.1474	-0.1135	-0.1154	$B_3 > B_4 > B_1 > B_2$
8	-0.1274	-0.1346	-0.1009	-0.1021	$B_3 > B_4 > B_1 > B_2$
9	-0.1164	-0.1231	-0.0884	-0.0902	$B_3 > B_4 > B_1 > B_2$
12	-0.0887	-0.0941	-0.0587	-0.0605	$B_3 > B_4 > B_1 > B_2$

Table 5

Impact of $\Gamma_{\mathcal{E}}$ under the IVCTSFHWG operator

Inputs of $\Gamma_{\mathcal{E}}$	Score values				Ranking
	B_1	B_2	B_3	B_4	
2	0.0646	0.0649	0.084	0.0825	$B_3 > B_4 > B_1 > B_2$
4	0.0129	0.0113	0.0377	0.0362	$B_3 > B_4 > B_1 > B_2$
5	-0.0071	-0.0096	0.0192	0.0177	$B_3 > B_4 > B_1 > B_2$
7	-0.0404	-0.0445	-0.0118	-0.0134	$B_3 > B_4 > B_1 > B_2$
8	-0.0546	-0.0595	-0.0253	-0.0269	$B_3 > B_4 > B_1 > B_2$
9	-0.0677	-0.0731	-0.0378	-0.03937	$B_3 > B_4 > B_1 > B_2$
12	-0.1017	-0.1086	-0.0705	-0.0720	$B_3 > B_4 > B_1 > B_2$

We can see from Table 4 and Table 5 that the overall ranking of alternatives remains consistent, with B_3 as the optimal alternative. As a result, the devised technique has a high degree of stability w.r.t. the $\Gamma_{\mathcal{E}}$ inputs.

7.2. Comparative Analysis

In this section, we compare our proposed scheme with several AOs to prove the efficiency and superiority of the recommended method. Ali *et al.* [36] initiated a MADM approach using T-spherical fuzzy AOs. Ullah *et al.* [31] studied T-spherical fuzzy Hamacher weighted averaging (T-SFHWA) and T-spherical fuzzy Hamacher weighted geometric (T-SFHWG) operators with application to the evaluation of the performance of search and rescue robots. We resolved the previously stated MADM problem by employing the methods of [32, 37], and the outcomes are exhibited in Table 6.

Table 6
 Ranking of the alternatives

Operators	Score values				Ranking result
	B_1	B_2	B_3	B_4	
IVCTSFHWA	0.0509	0.0423	0.0818	0.0806	$B_3 > B_4 > B_1 > B_2$
IVCTSFHWG	-0.3807	-0.3839	-0.3489	-0.3515	$B_3 > B_4 > B_1 > B_2$
CTSFWA [36]	-0.1164	-0.1231	-0.0884	-0.0902	$B_3 > B_4 > B_1 > B_2$
CTSFWG [36]	-0.0677	-0.0731	-0.0378	-0.03937	$B_3 > B_4 > B_1 > B_2$
T – SFHWA [31]	-0.0887	-0.0941	-0.0587	-0.0605	$B_3 > B_4 > B_1 > B_2$
T – SFHWG [31]	-0.1017	-0.1086	-0.0705	-0.0720	$B_3 > B_4 > B_1 > B_2$

According to Table 6, it becomes evident that the ranking orders acquired through these methods and the developed scheme are identical. This reveals that the designed MADM approach is valid. Also, the results are graphically plotted in Figure 1.

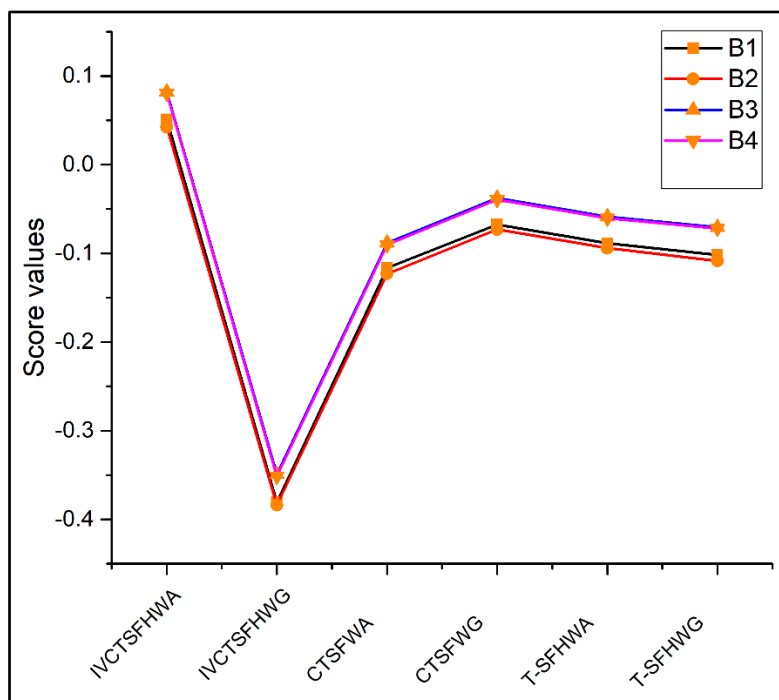


Fig. 1. Ranking of different methods

8. Conclusions

The IVCTSFS model is a generalization of TSFS and CFS. The Hamacher AOs are more flexible and powerful in many issues involving uncertainty, as they incorporate a regulatory parameter that can play a vital role in managing extreme values. Having a generic nature, both IVCTSFSs and Hamacher AOs require further study to be conducted. In this article, we established several Hamacher operational rules for IVCTSFNs. By using the Hamacher operational laws, we constructed a class of averaging and geometric AOs, namely the IVCTSFWA, IVCTSFHOWA, IVCTSFHWA, IVCTSFHWG, IVCTSFOWG, and IVCTSFHHWG operators. We also provided several cardinal features like idempotency, monotonicity, and boundedness. Meanwhile, we have designed a MADM method using the proposed operator.

To illustrate the proficiency and legitimacy of the developed method, we considered a MADM problem and solved it within the framed method. The influence of Hamacher variable parameters on the decision-making process is scrutinized, and the stability of ranking outcomes is deliberated. Lastly, a comparative analysis of rankings obtained using the projected and prevailing AOs underscored the significance of the designed methodology.

In our future work, we will extend our proposed work within the framework of complex neutrosophic hesitant fuzzy sets and spherical linear Diophantine fuzzy sets.

Appendix-1: Proof of Theorem 1

We will use the mathematical induction method to prove the desired result.

For $l = 2$, we have:

$$\Gamma_1 \mathcal{I}_1 \oplus \Gamma_2 \mathcal{I}_2$$

$$= \left[\begin{array}{l} \left[\begin{array}{l} \frac{\sqrt[Q]{\frac{(1+(N-1)\alpha_{a1}^Q)^{r_1} - (1-\alpha_{a1}^Q)^{r_1}}{(1+(N-1)\alpha_{a1}^Q)^{r_1} - (N-1)(1-\alpha_{a1}^Q)^{r_1}}}}}{\sqrt[Q]{\frac{(1+(N-1)\alpha_{a1}^Q)^{r_1} - (1-\alpha_{a1}^Q)^{r_1}}{(1+(N-1)\alpha_{a1}^Q)^{r_1} - (N-1)(1-\alpha_{a1}^Q)^{r_1}}}}} \cdot e^{2\pi i \frac{\sqrt[Q]{\frac{(1+(N-1)\alpha_{a1}^Q)^{r_1} - (1-\alpha_{a1}^Q)^{r_1}}{(1+(N-1)\alpha_{a1}^Q)^{r_1} - (N-1)(1-\alpha_{a1}^Q)^{r_1}}}}}{\sqrt[Q]{\frac{(1+(N-1)\alpha_{a1}^Q)^{r_1} - (1-\alpha_{a1}^Q)^{r_1}}{(1+(N-1)\alpha_{a1}^Q)^{r_1} - (N-1)(1-\alpha_{a1}^Q)^{r_1}}}}} \\ \frac{\sqrt[Q]{\frac{(1+(N-1)\alpha_{w1}^Q)^{r_1} - (1-\alpha_{w1}^Q)^{r_1}}{(1+(N-1)\alpha_{w1}^Q)^{r_1} - (N-1)(1-\alpha_{w1}^Q)^{r_1}}}}}{\sqrt[Q]{\frac{(1+(N-1)\alpha_{w1}^Q)^{r_1} - (1-\alpha_{w1}^Q)^{r_1}}{(1+(N-1)\alpha_{w1}^Q)^{r_1} - (N-1)(1-\alpha_{w1}^Q)^{r_1}}}}} \cdot e^{2\pi i \frac{\sqrt[Q]{\frac{(1+(N-1)\alpha_{w1}^Q)^{r_1} - (1-\alpha_{w1}^Q)^{r_1}}{(1+(N-1)\alpha_{w1}^Q)^{r_1} - (N-1)(1-\alpha_{w1}^Q)^{r_1}}}}}{\sqrt[Q]{\frac{(1+(N-1)\alpha_{w1}^Q)^{r_1} - (1-\alpha_{w1}^Q)^{r_1}}{(1+(N-1)\alpha_{w1}^Q)^{r_1} - (N-1)(1-\alpha_{w1}^Q)^{r_1}}}}} \end{array} \right], \\ \left[\begin{array}{l} \frac{\sqrt[Q]{\frac{\sqrt[Q]{(m_{i1})^{r_1}}}{(1+(N-1)(1-\alpha_{i1}^Q)^{r_1} + (N-1)\alpha_{i1}^Q)}}}{\sqrt[Q]{\frac{\sqrt[Q]{(m_{i1})^{r_1}}}{(1+(N-1)(1-\alpha_{i1}^Q)^{r_1} + (N-1)\alpha_{i1}^Q)}}}} \cdot e^{2\pi i \frac{\sqrt[Q]{\frac{\sqrt[Q]{(m_{i1})^{r_1}}}{(1+(N-1)(1-\alpha_{i1}^Q)^{r_1} + (N-1)\alpha_{i1}^Q)}}}{\sqrt[Q]{\frac{\sqrt[Q]{(m_{i1})^{r_1}}}{(1+(N-1)(1-\alpha_{i1}^Q)^{r_1} + (N-1)\alpha_{i1}^Q)}}}}} \\ \frac{\sqrt[Q]{\frac{\sqrt[Q]{(w_{i1})^{r_1}}}{(1+(N-1)(1-\alpha_{i1}^Q)^{r_1} + (N-1)\alpha_{i1}^Q)}}}{\sqrt[Q]{\frac{\sqrt[Q]{(w_{i1})^{r_1}}}{(1+(N-1)(1-\alpha_{i1}^Q)^{r_1} + (N-1)\alpha_{i1}^Q)}}}} \cdot e^{2\pi i \frac{\sqrt[Q]{\frac{\sqrt[Q]{(w_{i1})^{r_1}}}{(1+(N-1)(1-\alpha_{i1}^Q)^{r_1} + (N-1)\alpha_{i1}^Q)}}}{\sqrt[Q]{\frac{\sqrt[Q]{(w_{i1})^{r_1}}}{(1+(N-1)(1-\alpha_{i1}^Q)^{r_1} + (N-1)\alpha_{i1}^Q)}}}}} \end{array} \right], \\ \left[\begin{array}{l} \frac{\sqrt[Q]{\frac{\sqrt[Q]{(m_{n1})^{r_1}}}{(1+(N-1)(1-\alpha_{n1}^Q)^{r_1} + (N-1)\alpha_{n1}^Q)}}}{\sqrt[Q]{\frac{\sqrt[Q]{(m_{n1})^{r_1}}}{(1+(N-1)(1-\alpha_{n1}^Q)^{r_1} + (N-1)\alpha_{n1}^Q)}}}} \cdot e^{2\pi i \frac{\sqrt[Q]{\frac{\sqrt[Q]{(m_{n1})^{r_1}}}{(1+(N-1)(1-\alpha_{n1}^Q)^{r_1} + (N-1)\alpha_{n1}^Q)}}}{\sqrt[Q]{\frac{\sqrt[Q]{(m_{n1})^{r_1}}}{(1+(N-1)(1-\alpha_{n1}^Q)^{r_1} + (N-1)\alpha_{n1}^Q)}}}}} \\ \frac{\sqrt[Q]{\frac{\sqrt[Q]{(w_{n1})^{r_1}}}{(1+(N-1)(1-\alpha_{n1}^Q)^{r_1} + (N-1)\alpha_{n1}^Q)}}}{\sqrt[Q]{\frac{\sqrt[Q]{(w_{n1})^{r_1}}}{(1+(N-1)(1-\alpha_{n1}^Q)^{r_1} + (N-1)\alpha_{n1}^Q)}}}} \cdot e^{2\pi i \frac{\sqrt[Q]{\frac{\sqrt[Q]{(w_{n1})^{r_1}}}{(1+(N-1)(1-\alpha_{n1}^Q)^{r_1} + (N-1)\alpha_{n1}^Q)}}}{\sqrt[Q]{\frac{\sqrt[Q]{(w_{n1})^{r_1}}}{(1+(N-1)(1-\alpha_{n1}^Q)^{r_1} + (N-1)\alpha_{n1}^Q)}}}}} \end{array} \right] \end{array} \right]$$

$$\begin{aligned}
 & \left(\begin{aligned}
 & \left[\frac{\sqrt[Q]{\frac{(1+(N-1)m_{a_2}^Q)^{F_2} - (1-m_{a_2}^Q)^{F_2}}{(1+(N-1)m_{a_1}^Q)^{F_2} - (N-1)(1-m_{a_2}^Q)^{F_2}}}}}{\sqrt[Q]{\frac{(1+(N-1)m_{a_2}^Q)^{F_2} - (1-m_{a_2}^Q)^{F_2}}{(1+(N-1)m_{a_1}^Q)^{F_2} - (N-1)(1-m_{a_2}^Q)^{F_2}}}}}} \cdot e^{2\pi i \frac{Q}{\sqrt[Q]{\frac{(1+(N-1)\vartheta_{a_1}^Q)^{F_2} - (1-\vartheta_{a_1}^Q)^{F_2}}{(1+(N-1)\vartheta_{a_1}^Q)^{F_2} - (N-1)(1-\vartheta_{a_1}^Q)^{F_2}}}}} \right] \\
 & \left[\frac{\sqrt[Q]{\frac{(1+(N-1)u_{a_2}^Q)^{F_2} - (1-u_{a_2}^Q)^{F_2}}{(1+(N-1)u_{a_1}^Q)^{F_2} - (N-1)(1-u_{a_2}^Q)^{F_2}}}}}{\sqrt[Q]{\frac{(1+(N-1)u_{a_2}^Q)^{F_2} - (1-u_{a_2}^Q)^{F_2}}{(1+(N-1)u_{a_1}^Q)^{F_2} - (N-1)(1-u_{a_2}^Q)^{F_2}}}}} \cdot e^{2\pi i \frac{Q}{\sqrt[Q]{\frac{(1+(N-1)\beta_{a_1}^Q)^{F_2} - (1-\beta_{a_1}^Q)^{F_2}}{(1+(N-1)\beta_{a_1}^Q)^{F_2} - (N-1)(1-\beta_{a_1}^Q)^{F_2}}}}} \right] \\
 & \left[\frac{\sqrt[Q]{\frac{(\sqrt[Q]{N} m_{i_2})^{F_2}}{(1+(N-1)(1-m_{i_2}^Q))^{F_2} + (N-1)(m_{i_2}^Q)^{F_2}}}}}{\sqrt[Q]{\frac{(\sqrt[Q]{N} m_{i_2})^{F_2}}{(1+(N-1)(1-m_{i_2}^Q))^{F_2} + (N-1)(m_{i_2}^Q)^{F_2}}}}} \cdot e^{2\pi i \frac{Q}{\sqrt[Q]{\frac{(\vartheta_{i_2}^Q)^{F_2}}{(1+(N-1)(1-\vartheta_{i_2}^Q))^{F_2} + (N-1)(\vartheta_{i_2}^Q)^{F_2}}}}} \right] \\
 & \left[\frac{\sqrt[Q]{\frac{(\sqrt[Q]{N} u_{i_2})^{F_2}}{(1+(N-1)(1-u_{i_2}^Q))^{F_2} + (N-1)(u_{i_2}^Q)^{F_2}}}}}{\sqrt[Q]{\frac{(\sqrt[Q]{N} u_{i_2})^{F_2}}{(1+(N-1)(1-u_{i_2}^Q))^{F_2} + (N-1)(u_{i_2}^Q)^{F_2}}}}} \cdot e^{2\pi i \frac{Q}{\sqrt[Q]{\frac{(\beta_{i_2}^Q)^{F_2}}{(1+(N-1)(1-\beta_{i_2}^Q))^{F_2} + (N-1)(\beta_{i_2}^Q)^{F_2}}}}} \right] \\
 & \left[\frac{\sqrt[Q]{\frac{(\sqrt[Q]{N} m_{n_2})^{F_2}}{(1+(N-1)(1-m_{n_2}^Q))^{F_2} + (N-1)(m_{n_2}^Q)^{F_2}}}}}{\sqrt[Q]{\frac{(\sqrt[Q]{N} m_{n_2})^{F_2}}{(1+(N-1)(1-m_{n_2}^Q))^{F_2} + (N-1)(m_{n_2}^Q)^{F_2}}}}} \cdot e^{2\pi i \frac{Q}{\sqrt[Q]{\frac{(\vartheta_{n_2}^Q)^{F_2}}{(1+(N-1)(1-\vartheta_{n_2}^Q))^{F_2} + (N-1)(\vartheta_{n_2}^Q)^{F_2}}}}} \right] \\
 & \left[\frac{\sqrt[Q]{\frac{(\sqrt[Q]{N} u_{n_2})^{F_2}}{(1+(N-1)(1-u_{n_2}^Q))^{F_2} + (N-1)(u_{n_2}^Q)^{F_2}}}}}{\sqrt[Q]{\frac{(\sqrt[Q]{N} u_{n_2})^{F_2}}{(1+(N-1)(1-u_{n_2}^Q))^{F_2} + (N-1)(u_{n_2}^Q)^{F_2}}}}} \cdot e^{2\pi i \frac{Q}{\sqrt[Q]{\frac{(\beta_{n_2}^Q)^{F_2}}{(1+(N-1)(1-\beta_{n_2}^Q))^{F_2} + (N-1)(\beta_{n_2}^Q)^{F_2}}}}} \right]
 \end{aligned} \right) \\
 & = \left(\begin{aligned}
 & \left[\frac{\sqrt[Q]{\frac{\prod_{\xi=1}^2 (1+(N-1)m_{a_\xi}^Q)^{F_\xi} - \prod_{\xi=1}^2 (1-m_{a_\xi}^Q)^{F_\xi}}{\prod_{\xi=1}^2 (1+(N-1)m_{a_\xi}^Q)^{F_\xi} + (N-1) \prod_{\xi=1}^2 (1-m_{a_\xi}^Q)^{F_\xi}}}}}{\sqrt[Q]{\frac{\prod_{\xi=1}^2 (1+(N-1)m_{a_\xi}^Q)^{F_\xi} - \prod_{\xi=1}^2 (1-m_{a_\xi}^Q)^{F_\xi}}{\prod_{\xi=1}^2 (1+(N-1)m_{a_\xi}^Q)^{F_\xi} + (N-1) \prod_{\xi=1}^2 (1-m_{a_\xi}^Q)^{F_\xi}}}}} \cdot e^{2\pi i \frac{Q}{\sqrt[Q]{\frac{\prod_{\xi=1}^2 (1+(N-1)\vartheta_{a_\xi}^Q)^{F_\xi} - \prod_{\xi=1}^2 (1-\vartheta_{a_\xi}^Q)^{F_\xi}}{\prod_{\xi=1}^2 (1+(N-1)\vartheta_{a_\xi}^Q)^{F_\xi} + (N-1) \prod_{\xi=1}^2 (1-\vartheta_{a_\xi}^Q)^{F_\xi}}}}} \right] \\
 & \left[\frac{\sqrt[Q]{\frac{\prod_{\xi=1}^2 (1+(N-1)u_{a_\xi}^Q)^{F_\xi} - \prod_{\xi=1}^2 (1-u_{a_\xi}^Q)^{F_\xi}}{\prod_{\xi=1}^2 (1+(N-1)u_{a_\xi}^Q)^{F_\xi} + (N-1) \prod_{\xi=1}^2 (1-u_{a_\xi}^Q)^{F_\xi}}}}}{\sqrt[Q]{\frac{\prod_{\xi=1}^2 (1+(N-1)u_{a_\xi}^Q)^{F_\xi} - \prod_{\xi=1}^2 (1-u_{a_\xi}^Q)^{F_\xi}}{\prod_{\xi=1}^2 (1+(N-1)u_{a_\xi}^Q)^{F_\xi} + (N-1) \prod_{\xi=1}^2 (1-u_{a_\xi}^Q)^{F_\xi}}}}} \cdot e^{2\pi i \frac{Q}{\sqrt[Q]{\frac{\prod_{\xi=1}^2 (1+(N-1)\beta_{a_\xi}^Q)^{F_\xi} - \prod_{\xi=1}^2 (1-\beta_{a_\xi}^Q)^{F_\xi}}{\prod_{\xi=1}^2 (1+(N-1)\beta_{a_\xi}^Q)^{F_\xi} + (N-1) \prod_{\xi=1}^2 (1-\beta_{a_\xi}^Q)^{F_\xi}}}}} \right] \\
 & \left[\frac{\sqrt[Q]{\frac{\prod_{\xi=1}^2 (1+(N-1)m_{i_\xi}^Q)^{F_\xi}}{\prod_{\xi=1}^2 (1+(N-1)(1-m_{i_\xi}^Q))^{F_\xi} + (N-1) \prod_{\xi=1}^2 (m_{i_\xi}^Q)^{F_\xi}}}}}{\sqrt[Q]{\frac{\prod_{\xi=1}^2 (1+(N-1)m_{i_\xi}^Q)^{F_\xi}}{\prod_{\xi=1}^2 (1+(N-1)(1-m_{i_\xi}^Q))^{F_\xi} + (N-1) \prod_{\xi=1}^2 (m_{i_\xi}^Q)^{F_\xi}}}}} \cdot e^{2\pi i \frac{Q}{\sqrt[Q]{\frac{\prod_{\xi=1}^2 (\vartheta_{i_\xi}^Q)^{F_\xi}}{\prod_{\xi=1}^2 (1+(N-1)(1-\vartheta_{i_\xi}^Q))^{F_\xi} + (N-1) \prod_{\xi=1}^2 (\vartheta_{i_\xi}^Q)^{F_\xi}}}}} \right] \\
 & \left[\frac{\sqrt[Q]{\frac{\prod_{\xi=1}^2 (1+(N-1)u_{i_\xi}^Q)^{F_\xi}}{\prod_{\xi=1}^2 (1+(N-1)(1-u_{i_\xi}^Q))^{F_\xi} + (N-1) \prod_{\xi=1}^2 (u_{i_\xi}^Q)^{F_\xi}}}}}{\sqrt[Q]{\frac{\prod_{\xi=1}^2 (1+(N-1)u_{i_\xi}^Q)^{F_\xi}}{\prod_{\xi=1}^2 (1+(N-1)(1-u_{i_\xi}^Q))^{F_\xi} + (N-1) \prod_{\xi=1}^2 (u_{i_\xi}^Q)^{F_\xi}}}}} \cdot e^{2\pi i \frac{Q}{\sqrt[Q]{\frac{\prod_{\xi=1}^2 (\beta_{i_\xi}^Q)^{F_\xi}}{\prod_{\xi=1}^2 (1+(N-1)(1-\beta_{i_\xi}^Q))^{F_\xi} + (N-1) \prod_{\xi=1}^2 (\beta_{i_\xi}^Q)^{F_\xi}}}}} \right] \\
 & \left[\frac{\sqrt[Q]{\frac{\prod_{\xi=1}^2 (1+(N-1)m_{n_\xi}^Q)^{F_\xi}}{\prod_{\xi=1}^2 (1+(N-1)(1-m_{n_\xi}^Q))^{F_\xi} + (N-1) \prod_{\xi=1}^2 (m_{n_\xi}^Q)^{F_\xi}}}}}{\sqrt[Q]{\frac{\prod_{\xi=1}^2 (1+(N-1)m_{n_\xi}^Q)^{F_\xi}}{\prod_{\xi=1}^2 (1+(N-1)(1-m_{n_\xi}^Q))^{F_\xi} + (N-1) \prod_{\xi=1}^2 (m_{n_\xi}^Q)^{F_\xi}}}}} \cdot e^{2\pi i \frac{Q}{\sqrt[Q]{\frac{\prod_{\xi=1}^2 (\vartheta_{n_\xi}^Q)^{F_\xi}}{\prod_{\xi=1}^2 (1+(N-1)(1-\vartheta_{n_\xi}^Q))^{F_\xi} + (N-1) \prod_{\xi=1}^2 (\vartheta_{n_\xi}^Q)^{F_\xi}}}}} \right] \\
 & \left[\frac{\sqrt[Q]{\frac{\prod_{\xi=1}^2 (1+(N-1)u_{n_\xi}^Q)^{F_\xi}}{\prod_{\xi=1}^2 (1+(N-1)(1-u_{n_\xi}^Q))^{F_\xi} + (N-1) \prod_{\xi=1}^2 (u_{n_\xi}^Q)^{F_\xi}}}}}{\sqrt[Q]{\frac{\prod_{\xi=1}^2 (1+(N-1)u_{n_\xi}^Q)^{F_\xi}}{\prod_{\xi=1}^2 (1+(N-1)(1-u_{n_\xi}^Q))^{F_\xi} + (N-1) \prod_{\xi=1}^2 (u_{n_\xi}^Q)^{F_\xi}}}}} \cdot e^{2\pi i \frac{Q}{\sqrt[Q]{\frac{\prod_{\xi=1}^2 (\beta_{n_\xi}^Q)^{F_\xi}}{\prod_{\xi=1}^2 (1+(N-1)(1-\beta_{n_\xi}^Q))^{F_\xi} + (N-1) \prod_{\xi=1}^2 (\beta_{n_\xi}^Q)^{F_\xi}}}}} \right]
 \end{aligned} \right)
 \end{aligned}$$

This reveals that for $l = 2$ the result is valid.

Now, we can prove the result is true for $l = k \oplus 1$ and $l = k$.

- [7] Alkouri, A.M.D.J.S., & Salleh, A.R. (2012). Complex intuitionistic fuzzy sets. *AIP Conference Proceedings*, 1482(1), 464-470. American Institute of Physics.
- [8] Ullah, K., Mahmood, T., Ali, Z., & Jan, N. (2020). On some distance measures of complex Pythagorean fuzzy sets and their applications in pattern recognition. *Complex & Intelligent Systems*, 6, 15-27.
- [9] Khan, Q., Khattak, H., AlZubi, A.A., & Alanazi, J.M. (2022). Multiple Attribute Group Decision-Making Based on Intuitionistic Fuzzy Schweizer-Sklar Generalized Power Aggregation Operators. *Mathematical Problems in Engineering*, 2022(1), 4634411.
- [10] Liu, P., & Wang, P. (2018). Some q-rung orthopair fuzzy aggregation operators and their applications to multiple-attribute decision making. *International Journal of Intelligent Systems*, 33(2), 259-280.
- [11] Wang, J., Wei, G., Wang, R., Alsaadi, F. E., Hayat, T., Wei, C., et al. (2019). Some q-rung interval-valued orthopair fuzzy Maclaurin symmetric mean operators and their applications to multiple attribute group decision making. *International Journal of Intelligent Systems*, 34(11), 2769-2806. <https://10.1002/int.22156>.
- [12] Garg, H. (2017). Some picture fuzzy aggregation operators and their applications to multicriteria decision-making. *Arabian Journal for Science and Engineering*, 42(12), 5275-5290. <https://doi.org/10.1007/s13369-017-2625-9>.
- [13] Zhou, L., Chen, H., & Liu, J. (2012). Generalized power aggregation operators and their applications in group decision making. *Computers & Industrial Engineering*, 62(4), 989-999. <https://doi.org/10.1016/j.cie.2011.12.025>.
- [14] Gul, R., & Sarfraz, M. (2025). Enhancing Artificial Intelligence Models with Interval-Valued Picture Fuzzy Sets and Sugeno-Weber Triangular Norms. *Spectrum of Engineering and Management Sciences*, 3(1), 126-146. <https://doi.org/10.31181/sems31202540g>.
- [15] Darvesh, A., Naeem, K., & Shah, S. B. H. (2025). Scrutiny of Neutrosophic Cubic Fuzzy Data with Different Parametric Values using Bonferroni Weighted Mean: A Robust MCDM Approach. *Decision Making Advances*, 3(1), 245-265. <https://doi.org/10.31181/dma31202576>.
- [16] Ullah, F., & Waqar Shah, S. (2024). Optimizing Multi-attributive Decision-making: A Novel Approach through Matrix Theory and Fuzzy Hypersoft Sets. *Spectrum of Engineering and Management Sciences*, 2(1), 100-109. <https://doi.org/10.31181/sems2120248r>.
- [17] Klement, E.P., Mesiar, R., & Pap, E. (2004). Triangular norms. Position paper III: continuous t-norms. *Fuzzy Sets and Systems*, 145(3), 439-454.
- [18] Wu, L., Wei, G., Gao, H., & Wei, Y. (2018). Some interval-valued intuitionistic fuzzy Dombi Hamy mean operators and their application for evaluating the elderly tourism service quality in tourism destination. *Mathematics*, 6(12), 294. <https://10.3390/math6120294>.
- [19] Akram, M., Dudek, W.A., & Dar, J.M. (2019). Pythagorean Dombi fuzzy aggregation operators with application in multicriteria decision-making. *International Journal of Intelligent Systems*, 34(11), 3000-3019.
- [20] Wang, W., & Liu, X. (2011). Intuitionistic fuzzy geometric aggregation operators based on Einstein operations. *International Journal of Intelligent Systems*, 26(11), 1049-1075.
- [21] Riaz, M., Athar Farid, H. M., Kalsoom, H., Pamučar, D., & Chu, Y. M. (2020). A robust q-rung orthopair fuzzy Einstein prioritized aggregation operators with application towards MCGDM. *Symmetry*, 12(6), 1058.
- [22] Fahmi, A., Amin, F., Abdullah, S., & Ali, A. (2018). Cubic fuzzy Einstein aggregation operators and its application to decision-making. *International Journal of Systems Science*, 49(11), 2385-2397. <https://10.1080/00207721.2018.1503356>.
- [23] Senapati, T., Chen, G., & Yager, R. R. (2022). Aczel–Alsina aggregation operators and their application to intuitionistic fuzzy multiple attribute decision making. *International Journal of Intelligent Systems*, 37(2), 1529-1551. <https://10.1002/int.22684>.
- [24] Yang, Y., Chen, Z. S., Chen, Y. H., & Chin, K. S. (2018). Interval-valued Pythagorean fuzzy Frank power aggregation operators based on an isomorphic Frank dual triple. *International Journal of Computational Intelligence Systems*, 11(1), 1091-1110. <https://doi.org/10.2991/ijcis.11.1.83>.
- [25] Jia, Q., Hu, J., & Herrera-Viedma, E. (2021). A Novel Solution for $\$ Z \$$ -Numbers Based on Complex Fuzzy Sets and Its Application in Decision-Making System. *IEEE Transactions on Fuzzy Systems*, 30(10), 4102-4114. <https://10.1109/TFUZZ.2021.3138649>.
- [26] Wang, W., & Liu, X. (2012). Intuitionistic fuzzy information aggregation using Einstein operations. *IEEE Transactions on Fuzzy Systems*, 20(5), 923-938. <https://10.1109/TFUZZ.2012.2189405>.
- [27] Klement, E.P., Mesiar, R., & Pap, E. (2004). Triangular norms. Position paper II: general constructions and parameterized families. *Fuzzy Sets and Systems*, 145(3), 411-438.
- [28] Garg, H. (2019). Intuitionistic fuzzy hamacher aggregation operators with entropy weight and their applications to multi-criteria decision-making problems. *Iranian Journal of Science and Technology, Transactions of Electrical Engineering*, 43(3), 597-613.

- [29] Wu, S.-J., & Wei, G.-W. (2017). Pythagorean fuzzy Hamacher aggregation operators and their application to multiple attribute decision making. *International Journal of Knowledge-based and Intelligent Engineering Systems*, 21(3), 189-201. [https://10.1016/S0165-0114\(98\)00167-5](https://10.1016/S0165-0114(98)00167-5).
- [30] Darko, A.P., & Liang, D. (2020). Some q-rung orthopair fuzzy Hamacher aggregation operators and their application to multiple attribute group decision making with modified EDAS method. *Engineering Applications of Artificial Intelligence*, 87, 103259. <https://doi.org/10.1016/j.engappai.2019.103259>.
- [31] Ullah, K., Mahmood, T., & Garg, H. (2020). Evaluation of the performance of search and rescue robots using T-spherical fuzzy Hamacher aggregation operators. *International Journal of Fuzzy Systems*, 22(2), 570-582. <https://10.1007/s40815-020-00803-2>.
- [32] Xu, Y., & Wang, H. (2012). The induced generalized aggregation operators for intuitionistic fuzzy sets and their application in group decision making. *Applied Soft Computing*, 12(3), 1168-1179. <https://10.1016/j.asoc.2011.11.003>.
- [33] Wei, G., & Lu, M.A.O. (2017). Dual hesitant Pythagorean fuzzy Hamacher aggregation operators in multiple attribute decision making. *Archives of Control Sciences*, 27(3), 365-395. <https://10.1515/acsc-2017-0024>.
- [34] Ali, Z., Mahmood, T., Ullah, K., & Khan, Q. (2021). Einstein geometric aggregation operators using a novel complex interval-valued pythagorean fuzzy setting with application in green supplier chain management. *Reports in Mechanical Engineering*, 2(1), 105-134. <https://10.31181/rme2001020105t>.
- [35] Klement, E.P., & Navara, M. (1999). A survey on different triangular norm-based fuzzy logics. *Fuzzy Sets and Systems*, 101(2), 241-251. [https://10.1016/S0165-0114\(98\)00167-5](https://10.1016/S0165-0114(98)00167-5).
- [36] Ali, Z., Mahmood, T., & Yang, M. S. (2020). Complex T-spherical fuzzy aggregation operators with application to multi-attribute decision making. *Symmetry*, 12(8), 1311. <https://doi.org/10.3390/sym12081311>.